Noisy Prices and Inference Regarding Returns

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ABSTRACT

Temporary deviations of trade prices from fundamental values impart bias to estimates of mean returns to individual securities, to differences in mean returns across portfolios, and to parameters estimated in return regressions. We consider a number of corrections, and show them to be effective under reasonable assumptions. In an application to CRSP monthly returns, the corrections indicate significant biases in uncorrected return premium estimates associated with an array of firm characteristics. The bias can be large in economic terms, e.g., equal to 50% or more of the corrected estimate for firm size and share price.

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Some of the most frequently studied research questions in the field of Finance invoke comparisons of mean rates of return across securities and portfolios. Such comparisons lie at the heart of the vast empirical asset pricing literature, and are central to the estimation of firms' cost of capital. Beyond formal asset pricing tests, researchers have assessed relations between mean returns and a diverse array of firm attributes, such as the quality of corporate governance (Gompers, Ishii and Metrick (2003)), aggregate short selling and institutional ownership (e.g. Asquith, Pathak, and Ritter (2005), media coverage (Fang and Peress (2009)), success of customer firms (Cohen and Frazzini (2008)), and credit ratings (Avramov, Chordia, Jostova, and Philipov (2007)), to name just a few. Studies that compare returns on stocks of interest to those of designated "benchmark" securities, as in Barber and Lyon (1997) and Lakonishok and Lee (2001), also measure mean returns.

However, it is broadly recognized that the price data used to compute security returns contains noise attributable to market imperfections. Noise arises from microstructure frictions such as bid-ask spreads, nonsynchronous trading, discrete price grids, and the temporary price impacts of order imbalances. Noise can also arise due to changes in investor sentiment or other behavioral factors, in combination with limits to arbitrage. We use the term noise to mean any temporary deviation of price from underlying value. We discuss the economics of noisy prices more fully in the subsequent section.

The effects of noisy prices on empirical estimates of return volatility have been studied extensively in the "realized volatility" literature.[1] In contrast, the effects on estimates of mean returns and return premia have received less attention. The most prominent exception is Blume and Stambaugh (1983), who showed that zero-mean noise in prices leads to strictly positive bias in individual securities' mean returns, with the magnitude of the bias in each security's mean return approximately equal to the variance of the noise in the security's prices. Blume and Stambaugh (1983) also showed that cross-sectional mean returns to equal-weighted (EW) portfolios are upward biased by the cross-sectional average of the individual security biases. This implies that a comparison of mean returns across equal-weighted portfolios can be misleading. If portfolios are created by sorting on a variable correlated with the variance of noise, then the upward bias will be greater for the portfolio containing noisier securities, and the difference in mean returns across portfolios is biased. Biases attributable to noisy prices also arise in regression analyses. In particular, Asparouhova, Bessembinder and Kalcheva (2010) show that noisy prices lead to biases in intercept and slope coefficients obtained in any ordinary least squares (OLS) regression using rates of return as the dependent variable.

Many researchers who study security returns make no allowance for the potential effects of noisy prices. For example, examining papers published in only two premier outlets, *The Journal of Finance* and *The Journal of Financial Economics*, over a recent five year (2005–2009) interval, we were able to identify twenty four papers that report equal-weighted mean returns and compare them across portfolios.[2] In addition, dozens, if not hundreds, of published studies report results of OLS regressions using security or portfolio returns as the dependent variable, including cross-sectional "Fama-MacBeth" regressions and time-series "Factor-model" regressions. The implicit assumption in these studies is that any effects of noise in prices are small enough to be safely ignored. In part this may reflect a perception that noise-induced biases are likely to be important only in daily (or higher frequency) returns, and not in the monthly returns that are most frequently studied.

This paper assesses the effects of noisy prices on inferences regarding mean returns to individual securities and portfolios, and regarding return premia associated with stock characteristics. To illustrate the potential importance of the issue, we study monthly Center for Research in Security Prices (CRSP) returns from 1966 to 2009, and obtain uncorrected return premia estimates associated with a representative set of firm characteristics, including trading volume, share price, illiquidity, market-to-book ratio, and firm size. We then compare these estimates to those obtained after correcting for the effects of noisy prices. We consider several possible corrections, including the buy-and-hold method implemented by Blume and Stambaugh (1983) and Conrad and Kaul (1993), among others, and the gross-return-weighting method implemented in Asparouhova, Bessembinder and Kalcheva (2010). We also assess the effects of value-weighting returns, when weights are based on prior-month market values, and (since researchers often form value-weighted portfolios on an annual basis) based on prior-December market values.

As discussed more fully in Section II, each of these corrections equates to computing weighted average portfolio returns or estimating regression parameters by weighted least squares. The methods are distinguished by the weighting variable used, and the potential effectiveness of each method stems from the use of the lagged observed price in constructing weights. For brevity, and as explained in more detail in Section II, we refer to the buy-and-hold method implemented by Blume and Stambaugh (1983) as the "initial-equal-weighted" (IEW) method. We refer to the correction implemented by Asparouhova, Bessembinder and Kalcheva (2010) as the "return-weighted" (RW) method, and to weighting by the prior-period market capitalization as the "value-weighted" (VW) method. Finally, we refer to weighting by prior-December market value as the "annual-value-weighted" (AVW) method.

Blume and Stambaugh (1983) assume that the noise in security prices is independent across periods, i.e., that the noise in the period t price is on average dissipated by period $t + 1$. Asparouhova, Bessembinder and Kalcheva (2010) follow, and also assume that the noise in prices is independent across securities in their consistency proof. Here we assess, by theory and simulation, the effect of relaxing these assumptions. The results show that the corrected estimates are not necessarily consistent when the noise in prices is dependent across time or across securities. However, for any reasonable range of parameters, corrected estimates are strictly less biased than uncorrected (equal-weighted or OLS) estimates. That is, the effect of implementing the corrections considered is always to reduce the bias attributable to noisy prices. Further, for moderate violations of the independence assumption that are in line with the empirical estimates provided by Brennan and Wang (2010) and Hendershott, Li, Menkveld, and Seasholes (2011), the remaining bias in the corrected estimates is minimal.

In terms of effectiveness in mitigating biases in portfolio mean return estimates, the analysis provides little reason to prefer VW over RW, or vice versa. While each is effective in mitigating bias, the former places greater weight on large firms while the latter places essentially equal weight on each security in the sample. The final choice may therefore depend on researchers' preferences for weighting the information contained in the small versus large firms in the sample. In contrast, the VW method strictly dominates the AVW method in terms of mitigating the bias, which reflects that the AVW method does not correct for bias in months other than the first month after portfolio formation.[3] Further, for realistic parameters, the RW method contains less bias than the IEW method when estimating mean portfolio returns. Applied to the estimation of slope coefficients in regressions with returns as the dependent variable, we find for reasonable parameter estimates that the VW, RW and the IEW methods are all effective in mitigating the bias, and that the differences across corrected estimates are small.

Empirically, comparisons of uncorrected (EW or OLS) return premium estimates to estimates corrected by any of the weighting methods indicate statistically significant bias for all five firm characteristics considered, as well as for market $Beta$. However, the magnitude of the estimated bias varies considerably across characteristics. The RW and IEW estimates of the return premium associated with the book-to-market ratio differ by less than 10% from the uncorrected estimates, indicating modest bias. In contrast, the estimated biases in the return premia associated with firm size, share price, trading activity, and illiquidity are more substantial, equating to over 50% of the corrected estimates. While we focus here on return premia associated with five representative firm characteristics, similar biases potentially affect any variable used to explain average returns.

We delve further into the sources and economic interpretation of noise in prices in the next section. In Section II we assess the properties of the RW, IEW, and VW corrections, by theory and simulations. The empirical methodology and explanatory variables used are introduced in Section III. Empirical results are reported in Section IV, while Section V concludes.

I. The Economics of Noisy Prices

Numerous researchers have noted that the prices at which security trades take place can differ from underlying security values. We follow Blume and Stambaugh (1983) in referring to the underlying security value as the “true” price, and for simplicity of exposition we refer to the divergence of observed trade prices from true prices as “noise.” True prices have also been referred to as implicit or efficient prices, or fundamental values. The divergence of observed prices from true prices has also been referred to as mispricing, particularly when considering situations where the divergence is potentially larger than some traders’ costs of transacting. Regardless of whether one prefers the label noise or mispricing, if observed trade prices differ from true prices, then rates of return computed from observed prices differ from returns based on true prices. While some researchers may indeed want to make inferences regarding the characteristics of observed returns, we argue here that in many cases researchers will want to make inferences with regard to true returns, and we evaluate the properties of alternate methods of doing so.

A. Sources of Noise in Prices

In the discussion that follows we highlight the distinguishing characteristic of noise in transaction prices: noise is temporary, and is reversed over time. We interpret noise to mean any temporary deviation of transaction prices from true prices.

Scholes and Williams (1977), Blume and Stambaugh (1983), and Ball and Chordia (2001), among others, emphasize microstructure-based frictions such as bid-ask spreads, non-synchronous trading, and a discrete price grid as sources of noise in observed prices. Other authors have focused on the potentially important role of large orders or accumulated order imbalances. Grossman and Miller (1988) show that orders submitted by those who demand liquidity lead to price changes that are subsequently reversed on average, with the reversal compensating market makers for supplying immediate execution. Admati and Pfleiderer (1991) extend the Grossman and Miller model to allow outside speculators with fixed market participation costs to act as *de facto* market makers, who enter the market in response to large price movements caused by order imbalances. Bertsimas and Lo (1998) observe that short-term demand for even the most actively traded securities is not perfectly elastic, and develop a model of optimal execution strategies for large traders when their orders have both permanent and temporary effects on prices. Hasbrouck (2007, chapter 15) extends the analysis to allow for slowly decaying temporary price effects, where the transient price effects of order imbalances spill over into periods subsequent to order execution.

The microstructure-based literature implies that prices will generally contain noise even if all traders are fully rational. However, noise can also arise due to the presence of irrational traders. Black (1986) notes that "noise traders" include those with immediate liquidity needs, as well as traders who think they are informed, but are not. The vast "behavioral finance" literature (see, for example, the Barberis and Thaler (2003) survey) posits that some or all traders are not fully rational, e.g., because they do not update beliefs correctly, resulting in market prices that deviate from fundamental values. Models in which all traders exhibit behavioral biases or where barriers to arbitrage are sufficiently large can imply that prices are permanently altered, as compared to those implied by models of rational traders operating in frictionless markets. However, permanent deviations of price from value are likely not detectable by econometricians, and in any case impart no bias to either corrected or uncorrected return premium estimates, as shown in Section II.[4]

The empirical evidence generally confirms that order imbalances can temporarily push price away from value. Chordia and Subrahmanyam (2004) study individual New York Stock Exchange (NYSE) securities, and report that price changes are positively related to contemporaneous order imbalances, but negatively related to order imbalances over the four prior days. Andrade, Chang, and Seasholes (2008) study Taiwanese stocks, and also find a negative relation between price changes and prior-day order imbalances. In a study potentially important to researchers who focus on security prices measured at the monthly interval, Hendershott, Li, Menkveld, and Seasholes (2011) estimate that a quarter of the variance in monthly returns to NYSE stocks is due to transitory price changes that are themselves partially explained by cumulative order imbalances and measures of market-makers' inventories. Jegadeesh (1990) and Lehmann (1990) each document significant reversals of price changes for CRSP common stocks, the former at a one month horizon, the latter at a weekly horizon, also consistent with the notion that transaction prices contain significant noise.

Collectively, the literature implies that prices can differ from fundamentals values because cumulative order imbalances move prices if short-run liquidity supply is not perfectly elastic, and because not all traders are necessarily fully rational. If barriers are not too large the resulting divergence of observed from true prices can create opportunities for additional de facto liquidity providers to enter the market. The mechanism is well described by Harris (2003, page 414), who observes that "Large orders and cumulative order imbalances created by uninformed traders also cause prices to move from their fundamental values. The price changes reverse when value traders or arbitrageurs recognize that prices differ from fundamental values. Their trades then push prices back." With regard to the horizon over which noise is reversed, Harris (2003, page 414) notes that "The price impacts of large orders and order imbalances generated by uninformed traders may cause negative price change serial correlations measured over minutes, hours, days, or even months."

As noted, we interpret any temporary deviation of transaction prices from true prices as noise. However, not all temporary components in prices necessarily reflect noise. Poterba and Summers (1988), among others, have observed that time variation in required returns can induce a transitory component in prices. We assess in Internet Appendix to this paper whether the properties of the proposed corrections for noise are adversely affected by time variation in discount rates, and conclude that the effect on the corrections is minuscule for any reasonable parameterization.

B. Noisy Prices and Inference Regarding Price Appreciation

Let the observed period t price for any given stock be $P_t^o = P_t(1+\delta_t)$, where P_t denotes the true price. Ignoring dividends for simplicity, the true and observed (gross) period t returns are simply $R_t = P_t/P_{t-1}$ and $R_t^o = P_t^o/P_{t-1}^o$, respectively. We follow Brennan and Wang (2010) in relaxing the independence assumption to allow the noise component of the prices, denoted δ_t, to follow an AR(1) process. In Appendix A we show that the generalized version of the Blume and Stambaugh (1983) result regarding the expected return to any given security is:

$$E(R_t^o) \cong E(R_t)(1 + \sigma^2(1 - \rho)), \tag{1}$$

where σ^2 and ρ are the variance and first-order autocorrelation of δ_t respectively. The mean observed return is larger than the mean true return as long as $\rho < 1$, i.e., if the deviations of observed from true prices are indeed temporary. As in Blume and Stambaugh (1983) the differential between the mean observed and true returns increases with σ^2.

To illustrate the existence and the implications of noise-induced bias in mean observed returns, consider the following simple example. There are two securities, each of which has a constant true price equal to \$10. However, transaction prices for each security are affected by zero-mean noise. In particular, security 1 trades at a price of either \$9.9 or \$10.1, with equal probabilities. Security 2 is subject to more noise, and trades at either

\$9.8 or \$10.2, again with equal probabilities. The possible returns observed for security 1 are 2.02% (25% probability), -1.98% (25% probability) or zero (50% probability). For security 2, the possible observed returns are 4.08% (25% probability), -3.92% (25% probability) or zero (50% probability). In a large sample, the average return observed for security 1 will be 0.01%, while that observed for security 2 will be 0.04%.[5]

Notice that in this example neither the true price nor the expected observed price drifts upward over time. Nevertheless, positive mean returns are observed for both securities, and the mean is larger for the security with more noisy prices. The outcome that the average observed return overstates the rate of increase in prices is not specific to this example. The intuition remains intact when the true price, P_t, follows a random process (with or without drift) and for more complex noise distributions, as long as the noise is zero-mean. Consider for simplicity the case where true returns do not depend on past prices (as when true prices follow a martingale process), which implies that the expected true gross return at time t is the ratio of expected prices at time t and time $t-1$.[6] Also, given zero-mean noise, we have $E(P_t^o) = E(P_t)$ for any t. In combination we can write:

$$\frac{E(P_t^o)}{E(P_{t-1}^o)} = \frac{E(P_t)}{E(P_{t-1})} = E(R_t) \leq E(R_t^o) = E\left(\frac{P_t^o}{P_{t-1}^o}\right), \tag{2}$$

where the inequality results from expression (1).

Expression (2) implies that the growth rate in expected prices (true or observed) is strictly smaller than the expected observed return when prices contain noise.[7] The fact that the mean observed return overstates the rate at which prices trend upward over time comprises a key reason that many researchers will want to adjust observed returns for the effects of noisy prices. From expression (1), the divergence between the expected observed return and the growth in expected prices increases with the variance of the noise.

Note also that the value of investor holdings (aggregated across all agents in the economy) in any given firm is simply the number of shares outstanding times the price per share. The rate of growth in expected aggregate shareholder value is, for every firm, the same as the rate of growth in the expected share price. The implication is that researchers who are interested in studying the growth in expected shareholder value should focus on the expected true return.

C. Mean Observed Returns and Active Trading

The preceding discussion highlights what mean observed returns do not measure: the rate of growth in expected prices or aggregate shareholder wealth. We now turn to what mean observed returns do measure: returns to a hypothetical active trading strategy potentially used by a non-representative subset of investors.

Researchers who are interested in studying outcomes from active trading strategies will indeed want to study observed prices and returns (while making appropriate allowances for trading costs and other implementation issues). A focus on active trading can be motivated by the fact that *some* investors can potentially improve their returns by trading successfully on noise. As a case in point, Hsu (2006) shows that periodically rebalancing a portfolio to maintain equal weights can increase average portfolio returns relative to those earned on a value-weighted portfolio. His computations pertain in particular to an investor who succeeds in selling at prices that have increased (relative to other securities in the same portfolio), and vice versa, to reestablish equal weights. The strategy improves returns if the price changes that precipitated the trades are reversed on average, i.e., if prices contain noise. To the extent that the noise in prices reflects liquidity demand on the part of impatient traders, the posited rebalancing strategy is one of liquidity provision, and the improved returns can be interpreted as compensation for supplying liquidity.

However, if one investor sells (buys) at a price containing positive (negative) noise, another necessarily takes the opposite side of the trade. Gains and losses from active trading are zero-sum across all agents in the economy. The broader implication is that an equal-weighted cross-sectional mean of observed returns should be interpreted as pertaining to a hypothetical subset of investors who successfully execute an active rebalancing strategy. Similarly, the equal-weighted time-series mean observed return for an individual stock should be interpreted as pertaining to the hypothetical subset of investors who succeed in selling at prices that have increased and buying at prices that have decreased, so as to maintain constant dollar investment over time. In either case the experience of these hypothetical active investors does not reflect the experience of shareholders in the aggregate.

While our discussion to this point has focused on mean returns, the issues carry over to cross-sectional regressions with observed returns as dependent variable. Models such as the CAPM or the APT predict equilibrium pricing relationships, and do not explicitly allow for noise in prices. They therefore provide no explicit guidance as to whether researchers should use cross-sectional regressions to estimate parameters of the true or the observed return distribution.

We argue that the key cross-sectional implication of the CAPM and similar models is that positions taken in high risk (appropriately measured) securities should be associated with growth over time in the expected value of the position, relative to positions in low risk securities. In terms of the simple two-asset example in the prior subpsection, suppose that security 2 has more risk than security 1. Should the larger mean observed return for security 2 then be viewed as supportive of a positive risk-return tradeoff? Or, should the fact that expected prices for both securities are constant through time be interpreted to indicate the absence of return premia? We believe the latter interpretation is appropriate.

We summarize this discussion as follows. Researchers who are interested in studying growth in the aggregate value of all shareholdings in a given stock or group of stocks should conduct statistical inference with respect to the properties of true returns. In contrast, researchers who are interested in studying the potential profitability of specific active trading strategies will want to study observed prices and returns, while allowing for implementation issues. Researchers who report the equal-weighted mean observed return on individual stocks or portfolios, or coefficients estimated by OLS return regressions, are implicitly studying returns (before implementation costs) to active strategies potentially used by subsets of investors.

II. Return Estimators in the Presence of Noisy Prices

Researchers who wish to study the properties of true returns must still make their inferences on the basis of the noisy return data that is observable. In this section we discuss alternative methods of correcting cross-sectional parameter estimates for the effects of noisy prices. We then assess the large sample properties of those estimators, by theory and simulations.

A. The Weighted Estimators

We consider three main potential methods of correcting observed mean returns and return regression slope coefficients for the effects of noisy prices. The common thread across the three methods is that each involves weighting the observed time t return by a variable proportional to the time $t-1$ observed price. The intuition for the effectiveness of all three methods is conveyed by expression (13) in Blume and Stambaugh (1983), which shows that the expectation of a weighted portfolio return depends on expected weights, expected returns, and covariances between weights and returns. Expected observed returns are upward biased, as noted. A necessary condition for a weighting method to offset this bias is negative covariation between weights and observed returns. The use of a weighting variable proportional to the time $t-1$ observed price induces the requisite negative covariation: if the $t-1$ observed price contains positive noise then the weight is increased and the time t return is decreased, on average, and vice versa.

Weighting methods (including equal weighting and other constant-weight methods) that do not induce the requisite negative covariation will not eliminate the bias. In particular, mean returns to portfolios constructed on the basis of "fundamental" weighting (e.g., based on cash flows, dividends, or earnings) are, like equal-weighted portfolios, upward biased.

A.1. The Initial-Equal-Weighted Method

Blume and Stambaugh (1983) focus on cross-sectional mean portfolio returns, and introduce the "buy-and-hold" portfolio as a correction for noisy prices. The essential feature of a "buy-and-hold" portfolio is that the number of shares of each security is held fixed for some period of time. However, portfolio weights (the proportion of total investment in each security) depend both on share positions and prices. The weights in a "buy-and-hold" portfolio will therefore change through time as relative prices change. Conversely, to maintain constant portfolio weights requires changes in numbers of shares held to offset changes in share prices, highlighting that equal-weighted portfolios as well as other constant-weight portfolios imply active trading.

Blume and Stambaugh's theoretical motivation considered price-weighted portfolios, while their empirical analysis studied portfolios that are equal-weighted at the beginning of each calendar year, with share positions held constant through the subsequent year, before rebalancing to equal weights at year end. We refer to their empirical implementation as the "initially-equal-weighted" method, or IEW. Conrad and Kaul (1993) rely on the same method, but rebalance after three years.

The IEW method implies portfolio weights that evolve through time as a function of observed returns. In particular, if equal-weighted portfolios are formed at time zero, then the time t portfolio weighting variable for each stock n is $w_{nt} = \frac{P^0_{nt-1}}{P^0_{n0}}$, which reflects that securities with greater price appreciation subsequently receive greater weights in a non-rebalanced portfolio. Note, though, that the IEW method assigns equal weights to each security in the first period after the portfolio is formed ($t = 1$), implying the absence of any correction for the effects of noise in the first period.

A.2. The Return-Weighted Method

Asparouhova, Bessembinder and Kalcheva (2010) focus on cross-sectional return regressions, and implement a correction that involves weighted least squares estimation, with the prior-period gross return used as the weighting variable. Applied to estimating portfolio returns, this correction simplifies to computing a weighted mean return, where the weighting variable is the prior-period gross return.[8] We will denote this method RW.

Unlike IEW, the RW method does not have a "buy-and-hold" interpretation. The criterion that is assessed here is the ability of a method to provide consistent estimates of parameters of the true return distribution. The RW method can (under assumptions to be clarified) provide a consistent estimate of the mean true return on a single security, or of the mean true return to a portfolio of securities. The mean true return to an equal-weighted portfolio potentially differs (in particular if true returns are related to value) from the mean true return to a value-weighted portfolio. A researcher may well be interested in estimating the former, e.g., because value-weighted portfolios can be dominated by a few large capitalization stocks. Further, when estimating cross-sectional parameters, e.g., the return premium associated with beta or market capitalization, the information contained in the returns of a small capitalization stock is potentially as informative as that contained in the returns of a large capitalization stock, and the researcher may not want to weigh it less. Though it does not have a "buy-and-hold" interpretation, the RW method provides bias-corrected estimates of true mean returns and of true cross-sectional pricing parameters.

The RW method relies on the weighting variable $w_{nt} = R^0_{nt-1}$. To assess the relation between the RW method and the IEW method, consider a generalization where weighting is based on the prior s-period gross return: $w_{nt} = R^0_{nt-1-s} \ldots R^0_{nt-1} = \frac{P^0_{nt-1}}{P^0_{nt-1-s}}$, which we refer to $RW(s)$. Note that $RW(s)$ coincides with IEW when the relation between s and t (the number of periods since IEW portfolio formation) is $t - s = 1$. Thus, the IEW

method equates to $RW(1)$ in $t = 2$, to $RW(2)$ in $t = 3$, etc. In their empirical analysis Blume and Stambaugh compute weighted returns up to time $t = 12$, when the portfolio is once again rebalanced to equal weights. Estimates are averaged across months, and therefore are equivalent to the average across $RW(0)$ to $RW(11)$.

A.3. The Value-Weighted Method

As noted, Blume and Stambaugh's theoretical analysis focuses on price-weighted portfolios. While researchers rarely study price-weighted portfolios, they often study value-weighted portfolios. We consider VW when weights are based on prior-period market values, $w_{nt} = S_n P^0_{nt-1}$, where S_n is the number of shares outstanding for firm n. Note that, ignoring dividends, VW portfolios also reflect a "buy-and-hold" strategy. In addition, since researchers commonly form value-weighted portfolios on an annual basis, we consider the properties of a annual-value-weight (AVW) method that relies on prior December market values. Note though that, unlike IEW, RW, and VW, the AVW method does not rely on the time $t - 1$ price, except for the first period after portfolio formation.

B. The Framework

We next provide a formal assessment of the large sample properties of uncorrected and corrected estimates of mean returns and of cross-sectional regression slope coefficients. The true (gross) return for each security $n \in \{1, 2, ..., N\}$ at time $t \in \{1, 2, ..., T\}$ is assumed to be a linear function of observable variables,

$$R_{nt} = 1 + \alpha + \mathbf{X}'_{\mathbf{nt}}\boldsymbol{\beta} + \epsilon_{nt}, \tag{3}$$

where α is a scalar, $\boldsymbol{\beta}$ is a $K - 1$ -dimensional vector of parameters, ϵ_{nt} is a white noise random error term, and $\mathbf{X}_{\mathbf{nt}}$ is a $K-1$ -dimensional vector of firm parameters or market-wide factors.[9]

If $\mathbf{\tilde{X}_t} = (\mathbf{1}, \mathbf{X_t})$, where $\mathbf{1}$ is the N-dimensional vector of ones, $\mathbf{X_t} = (\mathbf{X_{1t}},\ \mathbf{X_{2t}}, \ldots, \mathbf{X_{Nt}})'$, and $\tilde{\boldsymbol{\beta}} = (1+\alpha,\ \boldsymbol{\beta}')'$, we can write the system of equations for the returns of firms 1 to N as

$$\mathbf{R_t} = \mathbf{\tilde{X}_t}\tilde{\boldsymbol{\beta}} = (1+\alpha)\mathbf{1} + \mathbf{X_t}\boldsymbol{\beta} + \boldsymbol{\epsilon}_\mathbf{t}. \tag{4}$$

In what follows, unless otherwise noted the expectation and covariance operators are applied cross-sectionally. Let μ_t denote the time-t true cross-sectional mean (gross) return, i.e., $\mu_t = 1 + \alpha + E(\mathbf{X'_{nt}}\boldsymbol{\beta})$. Observed prices, P^0_{nt}, deviate from "true" prices, P_{nt}: $P^0_{nt} = P_{nt}(1+\delta_{nt})$, where $\delta_{nt} = \sigma_n \delta^0_{nt}$, and δ^0_{nt} has a mean of zero and is independent of $(\mathbf{X_{m\boldsymbol{\tau}}},\ \sigma^2_n,\ \sigma^2_m)$ for any $n \neq m$ or $t \neq \tau$. We also assume that the noise variance parameters σ^2_n are draws from a common distribution across stocks, i.e., $\sigma^2_n \sim (\sigma^2, \Sigma)$ for all $n = 1,\ 2,\ \ldots,\ N$.

In the most general specification considered we allow:

$$\delta^0_{nt} = \rho\delta^0_{nt-1} + \sqrt{1-\rho^2}(\sqrt{c}\theta_t + \sqrt{1-c}\xi_{nt}), \tag{5}$$

where $E(\xi_{nt}) = E(\xi^3_{nt}) = 0$, $Var(\xi_{nt}) = Var(\theta_t) = 1$, and $E(\theta_t) = E(\theta^3_t) = 0$. This specification allows for potential serial correlation in noise through the ρ parameter, and allows for a potential common, market-wide, component in noise through the c parameter, while ensuring that the total variance of noise, σ^2_n, remains constant across ρ and c. If $c = 0$ the noise in prices is completely idiosyncratic, while if $c = 1$ there is no idiosyncratic component in the noise.

The observed (gross) return for stock n at time t is:

$$R^0_{nt} = R_{nt}\frac{1+\delta_{nt}}{1+\delta_{nt-1}} = R_{nt}D_{nt}, \tag{6}$$

where $D_{nt} = \frac{1+\delta_{nt}}{1+\delta_{nt-1}}$.

We evaluate the properties of weighted least squares estimators of the parameters, $\tilde{\boldsymbol{\beta}}$ of the linear specification (4) and of the time-t cross-sectional mean return, μ_t. With slight abuse of notation, $E(x_{nt})$ represents $plim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_{nt}$ for the random variable x_t. Similarly, $cov(x_{nt}, y_{nt})$, is the $plim_{N\to\infty}$ of $\frac{1}{N}\sum_{n=1}^{N}x_{nt}y_{nt} - \left(\frac{1}{N}\sum_{n=1}^{N}x_{nt}\right)\left(\frac{1}{N}\sum_{n=1}^{N}y_{nt}\right)$. We delineate the few cases where we refer to a time-series parameter by placing a bar over the relevant operator.

When estimating the cross-sectional mean (in the time t cross-section), the weighting scheme implies that $\mu_{WLS,t} = \frac{\sum_{n=1}^{N}w_{nt}R_{nt}^0}{\sum_{n=1}^{N}w_{nt}}$, equal to $\frac{E(w_{nt}R_{nt}^0)}{E(w_{nt})}$ in the limit. In a regression setting, when the weights w_{nt} are used in a weighted least squares (WLS) estimation of expression (4), the resulting (vector) parameter estimate is $\tilde{\boldsymbol{\beta}}_{\mathbf{WLS,t}} = [\mathbf{\tilde{X}_t'W_t\tilde{X}_t}]^{-1}\mathbf{\tilde{X}_t'W_tR_t^0}$. Here, $\mathbf{W_t}$ is a $N \times N$ diagonal matrix, with the weights w_{nt} on the diagonal. We are interested in the asymptotic properties of the estimator, or, in $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{WLS,t}} = [E(\mathbf{\tilde{X}_{nt}'\tilde{X}_{nt}}w_{nt})]^{-1}E(\mathbf{\tilde{X}_{nt}'}w_{nt}R_{nt}^0)$ and in how it compares to the estimator obtained from an OLS regression.

C. Properties of the Cross-sectional Estimators

We assess the asymptotic properties of estimators that rely on these weighting methods under a set of simplifying assumptions that allow for closed-form solutions. While the simplifying assumptions are somewhat restrictive, they convey the key intuition regarding the methods' effectiveness. We subsequently assess the effect of relaxing the simplifying assumptions by means of simulations. We initially focus on the case where the noise outcomes are independent across securities but potentially dependent through time. We then allow for cross-sectional commonality in noise realizations. All proofs regarding cross-sectional estimators are provided in Appendix B.

C.1. The Estimators when Noise is Independent Across Securities: $c = 0$

Mean Estimates

PROPOSITION 1: *If σ_n^2 and $\tilde{\mathbf{X}}_{\mathbf{nt}}$ are independent and $Cov(R_{nt}, R_{nt-1}) = 0$, then weighted cross-sectional averaging (in period t) yields estimates with the following properties (The approximations result from second-order Taylor series expansions.):*

- $plim_{N\to\infty}\ \mu_{EW,t} \approx \mu_t + \mu_t\sigma^2(1-\rho)$,
- $plim_{N\to\infty}\ \mu_{RW,t} \approx \mu_t + \mu_t\frac{\sigma^2(1-\rho)\rho}{1+\sigma^2(1-\rho)}$,
- $plim_{N\to\infty}\ \mu_{RW(s),t} \approx \mu_t + \mu_t\frac{\sigma^2(1-\rho)\rho^s}{1+\sigma^2(1-\rho^s)}$, *and*
- $plim_{N\to\infty}\ \mu_{VW,t} = \frac{E(P_{nt}S_n)}{E(P_{nt-1}S_n)}$.

Proof: See Appendix B.

Note that $Cov(R_{nt}, R_{nt-1})$ would be zero if either $\tilde{\boldsymbol{\beta}} = 0$ or if $\tilde{\mathbf{X}}_{\mathbf{nt}}$ does vary across n, either of which implies that expected returns are equal across n. Proposition 1 shows that the equal-weighted cross-sectional mean observed return is strictly upward biased, with the bias increasing in σ^2 and decreasing in ρ. The $RW(s)$ estimator of the cross-sectional mean is consistent if $\rho = 0$, but is upward biased if $\rho > 0$. Importantly, however, the $RW(s)$ bias is strictly smaller than the OLS bias for any s and ρ. Further, the bias in the $RW(s)$ estimator is decreasing in s.

If S_n is independent of prices, then $plim_{N\to\infty}\ \mu_{VW} = \frac{E(P_{nt})}{E(P_{nt-1})}$ and the VW estimate's bias depends on $Cov(R_{nt}, P_{nt-1})$. However, under the assumption of $Cov(R_{nt}, R_{nt-1}) = 0$, it follows that $Cov(R_{nt}, P_{nt-1}) = 0$ as well. Consistency of the VW estimate follows from the expression $E(R_{nt}) = \frac{E(P_{nt})}{E(P_{nt-1})} - \frac{cov(R_{nt}, P_{nt-1})}{E(P_{nt-1})}$.

Regression Estimates

PROPOSITION 2: *If σ_n^2 and $\tilde{\mathbf{X}}_{\mathbf{nt}}$ are independent, then application of WLS cross-sectional regression estimation (in period t) provides estimators with the properties:*

- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{OLS,t}} \approx \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)\tilde{\boldsymbol{\beta}}$,
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}} \approx \tilde{\boldsymbol{\beta}} + \frac{\sigma^2(1-\rho)\rho}{1+\sigma^2(1-\rho)}\tilde{\boldsymbol{\beta}}$,
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}} \approx \tilde{\boldsymbol{\beta}} + \frac{\sigma^2(1-\rho)\rho^s}{1+\sigma^2(1-\rho^s)}\tilde{\boldsymbol{\beta}}$,
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}} = \tilde{\boldsymbol{\beta}}$.

Proof: See Appendix B.

Proposition 2 shows that OLS regression coefficients are strictly biased in the direction of the true coefficients, with the bias increasing in σ^2 and decreasing in ρ. The $RW(s)$ regression coefficient estimator is consistent if $\rho = 0$, but is biased in the same direction as the OLS estimators if $\rho > 0$. Importantly, however, the $RW(s)$ bias is strictly smaller than the OLS bias for any s and ρ. With respect to ρ, $RW(s)$ achieves maximum bias at ρ close to $\frac{s}{s+1}$. The VW estimator is consistent for any ρ. Thus, under these assumptions all of the weighted estimators perform better than the OLS estimator, and the VW estimator performs best.

C.2. Allowing for Cross-sectional Commonality in Noise

We now consider the effect of allowing for $c > 0$, when the noise in prices is specified as in expression (5). We continue to assume that σ_n is independent of $\tilde{\mathbf{X}}_{\mathbf{nt}}$ for all n and all t. We further assume that the multivariate process $\tilde{\mathbf{X}}_{\mathbf{nt}}$ is stationary (e.g., $E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})$ does not depend on t.) Note that the period t estimator is conditional on the period t outcome on the common component of noise, θ_t, and thus need not be consistent. We therefore assess for each estimator the unconditional expectation (which we denote by $\bar{E}$) of the time-t cross-sectional $plim_{N\to\infty}$. Thus, $\bar{E}$ denotes the probability limit, $plim_{T\to\infty}$, of the time-series average of the cross-sectional $plim$'s. Because the estimates are functions of stationary random variables, unbiasedness of the cross-sectional $plim$'s implies sequential consistency of estimators obtained by time-series averaging of the cross-sectional estimates.[10] Also, let $\mu = \bar{E}(\mu_t)$.[11]

Mean Estimates

PROPOSITION 3: *If* $Cov(R_{nt}, R_{nt-1}) = 0$ *for all* t, *and* S_n *is independent of prices, then the application of weighted cross-sectional averaging leads to estimates with the following properties*

- $\bar{E}(plim_{N\to\infty}\ \mu_{EW,t}) \approx \mu + \mu\sigma^2(1-\rho)$,
- $\bar{E}(plim_{N\to\infty}\ \mu_{RW,t}) \approx \mu + \mu\sigma^2(1-\rho)(\rho + c(1-\rho))$,
- $\bar{E}(plim_{N\to\infty}\ \mu_{RW(s),t}) \approx \mu + \mu\sigma^2((1-\rho)\rho^s + c(1-2\rho^s+\rho^{s+1}))$,
- $\bar{E}(plim_{N\to\infty}\ \mu_{VW,t}) = \mu + \mu\sigma^2 c(1-\rho)$.

Proof: See Appendix B.

Proposition 3 shows that the bias in the equal-weighted cross-sectional mean return remains positive, and is unaffected by the degree of commonality in noise. All of the weighted estimators are adversely affected by commonality in noise, and none is consistent when c > 0. Blume and Stambaugh (1983) observed that their proposed correction is effective due to diversification of noise. The result here confirms this intuition, and shows that it applies to each of the corrections.

Importantly, the bias in the RW estimator of the cross-sectional mean return is strictly smaller than that of the OLS estimate as long as $|c + \rho - c\rho| < 1$. Focusing on the economically relevant cases where c and ρ range from 0 to 1, the RW estimator converges to the OLS estimate when either $c = 1$ (the noise is perfectly correlated across securities) or $\rho = 1$ (the noise in prices is permanent), but is otherwise strictly less biased than the OLS estimate. Under these assumptions the magnitude of the VW bias is always smaller than the $RW(s)$ bias.

Regression Estimates

PROPOSITION 4: *If $Cov(R_{nt}, R_{nt-1}) = 0$ then application of WLS regression leads to estimators with the following properties*

- $\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{OLS,t}}) = \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)\tilde{\boldsymbol{\beta}}$,
- $\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}}) = \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)(\rho + c(1-\rho))\tilde{\boldsymbol{\beta}}$,
- $\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}}) = \tilde{\boldsymbol{\beta}} + \sigma^2((1-\rho)\rho^s + c(1-2\rho^2+\rho^{s+1}))\tilde{\boldsymbol{\beta}}$,
- $\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}}) = \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)c\tilde{\boldsymbol{\beta}}$.

Proof: See Appendix B.

The properties of the regression parameters echo those of the cross-sectional means. In particular, OLS estimates of the regression parameters remain biased, and are unaffected by commonality in the noise. The time-series average of the weighted regression estimators are adversely affected by commonality in noise, and all are sequentially inconsistent when $c > 0$. The VW is again less biased than the OLS and the $RW(s)$ estimators. The RW estimator is less biased than the OLS estimator as long as $|c + \rho - c\rho| < 1$.

D. Relaxing the Restrictive Assumptions: Simulation-Based Evidence

The theoretical results reported in the previous section relied on restrictive assumptions, including an absence of cross-sectional variation in mean returns, and independence of the variance of noise from regression explanatory variables. We therefore assess the magnitude of potential biases and compare the performance of the proposed estimators using a simulation analysis that relaxes all of these assumptions to incorporate realistic parameters.

D.1. Calibration of the Simulation

Two of the most important parameters in terms of determining the magnitude of the biases attributable to noisy prices and the effectiveness of the proposed correction are the cross-sectional average noise standard deviation, σ, and the magnitude of cross-sectional variation in mean returns. With regard to the former, we rely on Brennan and Wang (2010), who provide what appears to be the most relevant evidence available to date. They study monthly returns to CRSP common stocks, and estimate $\sigma = 0.06$ (Brennan and Wang (2010, Table 2)). We assign a σ_n to each stock from a uniform distribution on $[0, 0.12]$.[12] We also accommodate dependence between σ_n and regression explanatory variables, again relying on estimates provided by Brennan and Wang. In particular, we choose parameters such that the correlation between σ_n and firm value is -0.235.[13]

We construct simulated true monthly returns according to

$$R_{nt} - 1 = \alpha + \beta_n(R_{mt} - 1) + \gamma_{illiq} I_n + \gamma_v V_{nt-1} + \epsilon_{nt}, \tag{7}$$

where R_m - 1 is the net market return, with mean equal to 1% and standard deviation of 5.5%, I_n is a (demeaned) measure of illiquidity, set equal to (σ_n - σ), and V_n is the (demeaned) market value of firm n. That is, we accommodate a market return premium as implied by the CAPM, as well as the empirical regularities that returns are related to illiquidity and to firm size. The standard deviation of firm specific returns, ϵ_{nt}, is set to 0.045.

Parameters are selected so that the standard deviation, across stocks, of the expected true monthly return is 1%. This parameter is potentially important, as the theoretical results above rely on the simplifying assumption that the cross-sectional covariance $Cov(R_{nt}, R_{nt-1})$ is zero.[14] Relaxation of this assumption potentially harms the properties of the weighted least squares estimators. We believe that a cross-sectional expected return standard deviation of 1% is on the high end of the range that could be considered realistic, as the two standard deviation range of expected returns varies from -1% to 3% per month.[15]

Given these parameters, we construct simulated true returns for each of the $N = 1000$

stocks, for periods $t = 0$ to 12, which accommodates evaluation of the IEW method with rebalancing after twelve months. For each true return, R_{nt}, we compute an observed return according to (6), when noise is specified according to expression (5). We iterate across values of ρ and c ranging from 0 to 0.9.

Having created simulated observed returns for periods $t = 1$ to 12, we estimate the cross-sectional mean return for each period based on EW, IEW, RW, and VW methods, and cross-sectional slope parameters by regressing the simulated observed returns on the market return, $illiq$, and lagged firm value by OLS, as well as by WLS using the IEW, RW, and VW weights. Mean returns and slope coefficients are averaged across the twelve periods and saved. The entire simulation is repeated 30,000 times, and we report averages across the 30,000 repetitions.

The properties of the OLS, RW, and VW methods are time invariant, but IEW properties are not. As noted, IEW is equivalent to equal-weighting in $t = 1$, and is equivalent to the $RW(t-1)$ method in subsequent periods. We report IEW results as the average across $t = 1$ to t =12, and also when period 1 is excluded.

D.2. Simulation Results

The key insight gained from the simulations is that the properties of the various estimators are generally consistent with the theoretical results derived above, despite the relaxation of various simplifying assumptions. We first discuss the properties of the regression slope coefficients estimated from the simulated observed returns. We focus our discussion on the estimation of β_{illiq}, as this coefficient is most directly affected by the noise in prices. Figure 1 displays the average difference between the slope coefficient estimated by different weighting methods and the true coefficient estimate, for ρ (the AR1 coefficient in the noise) ranging from 0 to 0.9.

<Figure 1 about here>

Panel A considers the case when c (the common component in noise) is fixed at zero. We observe that the bias in the coefficient estimated by OLS is largest (over 0.12, compared to a true coefficient of 0.15) when $\rho = 0$, and declines as ρ increases. The most important result observed on Panel A is that all three weighting methods provide estimates that are much less biased than OLS, for any ρ. In fact, the VW estimate is consistent in this case. The relative performance of the RW and IEW methods depends on the inclusion of period t =1. With the first period included, the IEW slope coefficient contains slightly more bias than RW. If the first period is excluded, the IEW slope contains slightly less bias (Figure 1(b)).

We note that the RW estimate of the slope coefficient is consistent when $\rho = 0$, but otherwise contains a small bias that achieves its maximum at $\rho = 0.5$. Importantly, though, even at the maximum, the bias in the RW estimate is less than one tenth as large as the unadjusted (OLS) estimate.

Panel B of Figure 1 presents estimated slope coefficients for ρ (the AR1 coefficient in noise) ranging from 0 to 0.9 and for c (the weight on the common component in noise) also ranging from 0 to 0.9. The results confirm that (i) the bias in the OLS slope coefficient is invariant to c, and (ii) all of the weighted estimates become more biased as c increases. Importantly, all of the weighted estimates remain strictly less biased than the OLS estimates, for any set of parameters.

Comparing across methods for correcting for bias, we observe that the VW method generally contains the least bias. The relative performance between RW and IEW depends on the inclusion of period 1 estimates in the IEW average. For completeness, Panel C presents the differences of the VW and IEW estimators from the RW estimator. We note, though, that differences across the corrected estimates are always small relative to the difference between the uncorrected (OLS) and the corrected estimates.

In summary, the simulations demonstrate that all three weighting methods provide large improvements over OLS estimation method when estimating regression slope coefficients. There is little meaningful economic difference across the corrected estimates, particularly if the first month is excluded from the IEW estimation.

We turn next to simulation results with regard to estimation of the cross-sectional mean return, displayed in Figure 2. We observe that the VW method provides downward biased estimates when $c = 0$ (Figure 2, Panel A). This reflects that the true returns incorporate a size effect, whereby the larger firms have lower expected returns. The RW method in this case gives estimates that are only slightly upward biased, and performs best overall. The IEW estimate of the cross-sectional mean return contains more bias than the RW method, even when period $t = 1$ is excluded, see Figure 2(b).[16]

<**Figure 2 about here**>

As was the case for regression slope coefficients, the bias in any of the corrected estimates of the cross-sectional mean return grows with c. Still, all of the corrected methods provide dramatic improvement over the equal-weighted estimation (except for when both ρ and c are close to 1). Also, as shown in Figures 2(c) and 2(d), differences between the RW, VW, and IEW estimates are generally not economically meaningful.

The available empirical evidence indicates that the degree of persistence in the noise in prices is modest. Evaluating monthly returns to CRSP common stocks, Brennan and Wang (2010) report a cross-sectional mean ρ estimate equal to 0.07, while Hendershott, Li, Menkveld, and Seasholes (2011) study monthly returns to NYSE stocks and report a mean estimate of 0.15. Given $c = 0$, the biases in the corrected estimates are small in any case, and are very close to zero for ρ in this range. We conclude from this analysis that the corrected measures (VW, RW, and IEW) are in this case robust to the potential existence of autocorrelation in the noise contained in prices, and provide estimates that are essentially free of bias.

In contrast, commonality in noise, $c > 0$, potentially affects the corrected estimates more substantively. Unfortunately, we are not aware of any direct empirical estimates of the degree of commonality across stocks in the noise component of prices.[17] This analysis supports the conclusion that uncorrected (EW or OLS) return premium estimates contain substantial biases that can be mitigated by the corrections discussed here. However, if the noise in prices contains a substantial common component, then the methods considered here only partially correct for the biases.

III. Data Description and Anticipated Effects of Noise

To assess the empirical relevance of biases attributable to noisy prices and the effect of implementing the proposed corrections, we study five firm-level explanatory variables that are broadly representative of those examined in the empirical asset pricing literature: firm size, trading volume, illiquidity, share price, and the book-to-market ratio. This analysis should be viewed as illustrative that the potential biases attributable to noisy prices are large enough to matter. The effects of implementing the corrections in other empirical applications are yet to be assessed. We study monthly returns in excess of the treasury interest rate for U.S. equities using CRSP data and the Compustat Industrial North America files. The sample spans the period July 1963 through December 2009, and consists of common stock (CRSP shrcd=10, 11 and 12) of NYSE-, Amex- and Nasdaq-listed companies (CRSP exchcd = 1, 2 and 3). The analysis of monthly returns considers the period January 1966 to December 2009, as the earliest sample months are used to construct systematic risk estimates.

A. Anticipated Direction of Bias

Asparouhova, Bessembinder, and Kalcheva (2010) show that the key determinant of the direction of the bias in uncorrected estimates of return premia is the sign of the cross-sectional covariation between the variance of the noise in prices, σ_n, and the firm attribute considered. To the extent that researchers can estimate or form conjectures regarding the sign of this covariance, the direction of the return premium bias can be anticipated.

Blume and Stambaugh (1983) have shown that biases attributable to noisy prices impart downward bias to the empirically negative relation between firm size and returns. This result is anticipated if the prices of small firms contain more noise, on average. Black (1986) conjectures that low share prices will be associated with substantially more noisy prices. If so, we predict a positive bias in estimates of the relation between returns and inverse share price. Empirical measures of illiquidity are likely to be strongly positively related to the variance of noise in prices, implying upward bias in associated return premium estimates. Trading volume is often interpreted as a measure of liquidity, and should therefore be negatively correlated across stocks with the variance of noise in prices. We conjecture that both market value of equity and book value of equity are negatively correlated across stocks with the variance of noise in prices. We therefore do not offer a prediction as to the sign of the covariance between the market-to-book ratio and noise, or of the possible bias in estimates of the "value premium."

B. Variable Construction

We consider five firm-level explanatory variables that are broadly representative of those examined in the empirical asset pricing literature. The following variables are constructed:

- *Size* - the natural logarithm of the market value of the equity of the firm as of the end of the second to last month.
- *log(BM)* - the natural logarithm of the ratio of the book value of equity plus deferred taxes to the market value of equity, using the end of the previous year market and book values.[18]
- *Dvol* - the natural logarithm of the dollar volume of trading in the security in the

second to last month.[19]

- *InvPrice* - the natural logarithm of the reciprocal of the share price as reported at the end of the second to last month.

- *Illiq* - the Amihud (2002) illiquidity measure, computed as the ratio of daily absolute return to daily dollar volume multiplied by 1,000,000, and averaged over all days with nonzero volume in the previous year. *Illiq* and *Dvol* are standardized as per Eq. 3 and Eq. 4 in Amihud (2002).

We include in the sample for a given month those stocks that satisfy the following criteria: (i) return data for the current month of December and in 24 of the previous 60 months is available on CRSP, and (ii) data is available to calculate the market capitalization, share price, and dollar volume as of the previous month. Following Fama and French (1992), we exclude financial firms from our sample. Nasdaq stocks generally enter the sample in 1983, due to the requirement that trading volume data be available. Firms are assigned to portfolios based on attributes measured as of end of the prior July. Following Brennan, Chordia, and Subrahmanyam (1998), firm-level explanatory variables are expressed as deviations from their monthly cross-sectional mean. We also include market *Beta* as a measure of risk in our regression analysis.[20]

Panel A of Table I reports the time-series averages of the cross-sectional means, medians and standard deviations for a number of key empirical variables, before log transformations, for the full sample. The mean monthly return is 0.827%. The mean market capitalization for the sample stocks is $1.182 Billion. Firm size, share price, illiquidity, and trading volume exhibit positive skewness, as evidenced by means that substantially exceed medians.

<Table I about here>

Panel B of Table I reports time-series averages of the monthly cross-sectional correlations. The largest correlations are between *Size* and *Dvol* (0.886) and *Size* and *InvPrice* (-0.783). The correlation of *Illiq* with *Size* and *Dvol* is -0.323 and -0.339, respectively. Firm size and the book-to-market ratio exhibit a substantial negative average correlation (-0.287), implying that firms that are small in absolute market capitalization tend to also be small relative to the book value of their assets.

IV. Empirical Results

A. Returns to Attribute-Sorted Portfolios

In this subsection we assess the effect of noisy prices on return premium estimates obtained by the common method of comparing mean returns across attribute-sorted portfolios. The portfolio returns are weighted based on the variables discussed previously. Table II reports mean returns to the first and tenth decile portfolios, and to the "hedge portfolio" that is long the tenth portfolio and short the first portfolio, for the five firm-level explanatory variables.

<Table II about here>

We focus on univariate portfolio sorts because it is possible to form reasonably strong conjectures as to the likely correlation, and hence the direction of noise-induced bias, between the variance of the noise in prices and individual explanatory variables. We subsequently report univariate and multivariate regression results that include various combinations of firm characteristics as regressors.

A.1. Equal-Weighted Portfolio Returns

Mean portfolio returns obtained when security returns are weighted equally are consistent with the findings of previous studies. Focusing on the column labeled $10-1$ for the mean returns to hedge-portfolios, we observe the well documented "size effect" (mean hedge portfolio return is 1.425% per month with an associated t-statistic of 4.43), and the "value premium" (mean hedge portfolio return of 1.369% with associated t-statistic of 6.03). For illiquidity-sorted portfolios the hedge portfolio return is 1.139% per month, with t-statistic of 3.95. Consistent with the regression-based results reported by Brennan, Chordia, and Subrahmanyam (1998), we observe a strong share price effect, as returns to the hedge portfolio (low share price decile less high share price decile) are positive and significant (1.252% per month, t-statistic = 3.27). Also consistent with their results, we observe a trading volume effect, as the mean return to the hedge portfolio that is long high-volume stocks and short low-volume stocks is -1.198% per month, with a t-statistic of -4.52.

A.2. Adjusting For Biases Due to Noise in Prices

The main focus of this paper is on correcting empirical estimates for the effects of noisy prices through the use of appropriate weighting methods. Since hedge portfolio returns provide evidence as to whether a given attribute is associated with cross-sectional variation in mean returns, we mainly discuss the difference in hedge portfolio returns across weighting methods, and present the results in a matrix in the center columns of each Panel of Table II. For each variable of interest, we compute the differential in mean hedge portfolio returns across all pairs of weighting variables. For example, on Panel A of Table II, the value -0.463% in the column labeled *RW* and row labeled *EW* is the difference in the hedge portfolio return obtained by the *RW* method (-0.961%) and that obtained by the *EW* method (-1.425%). A corresponding hedge portfolio differential is reported for each pair of weighting methods, along with associated t-statistics for the hypothesis that the associated differential is zero.

The key finding that can be observed in Table II is that every return premium estimated on the basis of equal-weighted portfolio returns is larger (in absolute magnitude) than any of the corrected estimates of the corresponding premium. The differentials in the estimated premia obtained by EW as compared to any of the corrected estimates are, with but a single exception, uniformly highly statistically significant, as evidenced by the t-statistics in the rows labeled EW on the right side of Table II.[21] Of particular interest are differentials across EW and RW, since each pertain to equal-weighted mean returns, the former uncorrected and the latter corrected for bias. T-statistics for the $EW - RW$ differential range in absolute value from 2.21 (for the book-to-market ratio) to 15.04 (for inverse share price). T-statistics for the $EW - VW$ differential are also large, but it should be noted that this differential reflects both the effect of removing bias and the shift to placing more weight on larger stocks.

The economic relevance of the bias attributable to noisy prices varies substantially across explanatory variables. The bias is quite relevant for firm size, share price, trading volume, and illiquidity. The bias is least relevant for the book-to-market ratio. Focusing in particular on the differential between EW and RW mean returns, Panel A shows that noise in prices explains about one third of the apparent size effect in monthly returns, as the estimated bias is -0.46% per month, (t-statistic = -13.52), compared to an equal-weighted hedge portfolio return of -1.43% per month. In contrast, results reported on Panel B indicate only a modest upward bias in the equal-weighted estimate of the "value premium." While the t-statistic for equal-weighted less return-weighted hedge portfolio return is significant, the economic magnitude of the bias estimate, 0.09% per month, is small relative to the estimated premium of 1.37% per month.

Panel C shows that noise in prices is particularly relevant for the apparent return premium associated with share price, as the bias ($EW - RW$) is estimated to be 0.61%

per month (t-statistic = 15.04), half as large as the apparent return premium of 1.25% per month. Share prices are not randomly distributed across stocks, but are influenced by managers' strategic choices, including IPO offer prices and stock split policy. It would therefore represent something of a puzzle if return premia were related to share prices (as implied by the EW t-statistic of 3.27), as it would suggest that firms could reduce the return premium and their cost of capital by altering share price. We observe that none of the bias-adjusted estimates support the existence of a return premium associated with share price, as the bias-corrected hedge portfolio t-statistics range from 0.40 (VW) to 1.74 (RW).

The results in Panel D indicate that the apparent relation between returns and trading activity is also partially attributable to noise in prices. The bias, based on the $EW - RW$ differential, is estimated at -0.35% per month, which comprises about a third of the uncorrected estimated premium associated with trading activity. Finally, Panel E shows that the magnitude of the upward bias in the estimate of the return premium for *Illiq* is considerable, as return-weighted hedge portfolio returns exceed equal-weighted hedge portfolio returns by 0.36% per month (t-statistic = 12.77).

Weighting by $t - 1$ Value

As noted, shifting from EW to VW entails two distinct effects: removal of bias due to noisy prices, and the shift to weighting large stocks more heavily. The $EW - VW$ differential in mean returns can be decomposed into the $EW - RW$ differential, which entails only the removal of bias, and the $VW - RW$ differential, which is an estimate of the effect of the shift in weights alone. Focusing for example on the estimated return premium associated with firm size, the $EW - VW$ return differential of -0.91% per month can be decomposed into the $EW - RW$ differential (removal of bias) of -0.46% and the $VW - RW$ (firm-size weighting effect) of -0.45% per month. Similar conclusions apply for all five firm characteristics, in that the effect of removing bias (the $EW - RW$ differential) is always significant (absolute t-statistics range from 2.21 for the book-to-market ratio to 15.04 for inverse share price), while the pure weighting effect (the $VW - RW$ differential) is also always significant (absolute t-statistics range from 2.05 for book-to-market ratio to 5.14 for firm size.)

Return premia estimates obtained by the IEW method are broadly similar to those obtained by the RW method. The IEW and RW hedge portfolio returns are uniformly smaller in absolute magnitude as compared to the unadjusted EW estimates, reflecting that both methods are largely effective in mitigating the bias due to noisy prices. The IEW and RW hedge portfolio returns are uniformly larger than the VW estimate, which reflects that IEW and RW both estimate the true equal-weighted, rather than value-weighted, mean return.

Annual Value-Weighting

Finally, the differential in VW vs. AVW hedge portfolio returns is of interest. The VW and AVW methods both weight large-capitalization securities more heavily. To the extent that large firms tend to have less noisy prices the effect will be to mitigate the bias attributable to noise. However, as noted, the key to eliminating the bias due to noisy prices is to use a weighting variable that includes the time $t-1$ share price. The VW, RW, and IEW methods do so. The AVW method does so only for January returns.

Comparing mean returns across the VW and AVW methods as reported on Table II, we see that the differential is minimal in the case of large firms (size portfolio 10), high-share-priced firm (inverse price portfolio 1), high-trading-volume firms (volume portfolio 10), and liquid firms (illiquidity portfolio 1). In contrast, substantial differentials in mean returns are observed across the VW and AVW methods for small, low-priced, illiquid, and low volume portfolios. For example, the mean VW return for size portfolio 1 is 0.89%, while the mean AVW return for the same portfolio is 1.24%. Since the estimates are based on the same stocks over the same time intervals, and each places greater weight on large firms, we conclude that the return differential is attributable to the failure of the AVW method to eliminate bias attributable to noisy prices in months other than January.

Since the AVW method allows bias to remain in the mean return to the least liquid (or lowest priced) portfolio, the hedge portfolio return estimated using AVW remains biased. While the $VW - AVW$ hedge portfolio differential is not statistically significant for the illiquidity ratio (t-statistic = -0.98) and is only marginally significant for the book-to-market ratio (t-statistic = 1.73), the differential is significant in the case of firm size (t-statistic = 5.40), inverse price (t-statistic = -2.03), and trading volume (t-statistic = 4.09). The implication is that researchers who wish to eliminate the effect of noisy prices by use of value-weighting should weight returns by time $t-1$ value, not by value measured at an earlier date when portfolios are formed.

A.3. Further Analysis and Robustness

The Effect of Excluding Low-Priced Securities

Some authors, including Jegadeesh and Titman (2001), Amihud (2002), and Pástor and Stambaugh (2003) mitigate the effects of noisy prices by excluding relatively illiquid securities (in particular those with low share prices) from their analyses. In Table III we report portfolio mean returns after excluding stocks with share price less than $5 as of the end of the preceding month.

<Table III about here>

The results indicate that eliminating low-priced securities is very effective in reducing the bias attributable to noisy prices. Comparing EW hedge-portfolio mean returns across Table II and Table III, we observe that the elimination of low-priced stocks always reduces the absolute magnitude of the EW hedge portfolio return, and that the reduction is always statistically significant. The magnitude of the reduction in bias is large. For example, for firm size the $EW - RW$ hedge portfolio return differential is reduced from 0.46% per month without the price filter to 0.06% per month with the price filter. Similarly, for inverse share price, the $EW - RW$ hedge portfolio return differential is reduced by the price filter from 0.61% per month to 0.06% per month. However, despite the reduction, return premia estimated by EW remain biased away from zero for every explanatory variable except the book-to-market ratio. Absolute t-statistics for the $EW - RW$ hedge portfolio differential on Table III range from 4.97 for illiquidity to 6.31 for firm size.

While the results reported on Table III support the conclusion that eliminating low-priced stocks from the sample substantially reduces the bias in EW estimates attributable to noisy prices, they also indicate a hidden cost of doing so. In particular, inference regarding the existence, magnitude, and functional form of return premia is substantially affected. Bias-adjusted hedge portfolio returns reported on Table III are uniformly smaller in absolute magnitude as compared to corresponding estimates on Table II, and the differentials across tables are often statistically significant. For example, the RW hedge portfolio return associated with firm size reported on Panel A of Table II is -0.96% per month with an associated t-statistic of -3.07, compared to a corresponding estimate of -0.29% per month with a t-statistic of -1.62 on Table III. Similar effects are observed on Panel D with respect to the bias-adjusted (RW) estimates of the hedge portfolio return for trading volume, which are -0.85% per month (t-statistic = -3.28) without the price filter, versus -0.38% per month (t-statistic = -2.09) with the price filter.

We conclude that a hidden cost of reducing noise-related bias by excluding low-priced stocks is the loss of valuable information regarding actual return premium contained in those stocks. Further, the lost information includes indications that the return premia are not linear in the attributes, as evidenced by substantial reductions in the absolute magnitude of the hedge portfolio returns, not just reductions in statistical significance due to a smaller sample size.

Bias-Adjustment and the Carhart-Fama-French Four Factor Model

Empirical results reported to this point have focused on raw returns (in excess of Treasury interest rates), and thus have not made any allowance for known sensitivities of returns to market-wide risk factors. We next estimate for each decile portfolio "*alphas*" (intercepts), when portfolio returns are computed on an EW, RW, and VW basis, by regressing portfolio returns on the Carhart-Fama-French factors (Carhart, 1997). This analysis allows examination of two issues. First, we can assess whether key results with regard to the effects of adjusting for noisy prices are sensitive to allowances for return sensitivity to the four factors. Second, we can assess whether security returns that have been adjusted for biases attributable to noisy prices are consistent with the implications of the four factor model.

Table IV reports alphas for portfolios 1, 10, and the $10-1$ hedge portfolio. The results indicate that adjusting portfolio returns for sensitivity to the Carhart-Fama-French four factors has essentially *no* effect on the magnitude of the various biases attributable to noise in prices. In particular, the bias estimates and associated t-statistics for the $EW - RW$ differential are uniformly little altered when focusing on alphas as compared to mean returns reported in Table II.

<**Table IV about here**>

The adjustment of returns for sensitivity to the Carhart-Fama-French factors does reduce the magnitude of some return regularities that survive the correction for noise in prices. Focusing on the bias-corrected results (RW) in the column labeled $10 - 1$, we observe that alphas are meaningfully closer to zero as compared to mean returns in the case of firm size, book-to-market ratio, and trading volume. In the case of inverse share price and illiquidity ratio, the alpha estimate is statistically indistinguishable from zero, indicating that the combination of the bias adjustment and allowance for sensitivity to the Carhart-Fama-French factors has eliminated the apparent return premium contained in the $10 - 1$ hedge portfolio.

To provide a more rigorous test of the hypothesis that the Fama-French-Carhart four factor model explains the cross-section of bias-adjusted returns, we report also the p-value obtained when implementing the F-test of Gibbons, Ross and Shanken (1989). This statistic pertains to the hypothesis that the regression intercepts for all ten attribute-sorted portfolios are simultaneously zero.

The resulting p-values indicate rejection of the four-factor model for all five firm attributes, both when portfolios are EW and RW. The former indicates that the four-factor model fails to fully explain equal-weighted portfolio returns. The latter indicates that the existence of noise-related bias in the EW returns is not the sole explanation, as the data continues to reject the model even when the equal-weighted returns are adjusted for bias attributable to noisy prices. Notably, however, the p-values do not indicate rejection of the four-factor model for any of the five attributes when firms are weighted by prior-period value. We conclude that the four-factor model can explain bias-adjusted returns to attribute-sorted portfolios, but only when the information contained in returns to smaller stocks is deemphasized by means of value-weighting.

January vs. Non-January Months

Numerous studies have documented return anomalies and/or strengthened empirical relations in the month of January. For example, Eleswarapu and Reinganum (1993) find a statistically significant relation between average return and bid-ask spread for NYSE stocks *only* in January, while Keim (1983) shows that the return premium associated with firm size is much stronger in January than in other months. Table V reports mean returns to attribute-sorted portfolios on an EW, RW, and VW basis, separately for the month of January and for non-January months.

<**Table V about here**>

The broadest observation regarding the mean return differentials reported on Table V is that *every* empirical relation is stronger in January than in other months. In particular the biases contained in EW returns attributable to noisy prices are uniformly larger in January. In the case of firm size, for example, the bias (estimated by the $EW - RW$ hedge portfolio differential) is -1.36% in January versus -0.38% in non-January months. The larger bias in January months could reflect that prices contain more noise in January, or that the cross-sectional correlation between noise and firm attributes is increased in January. However, the bias due to noisy prices is not confined to January. With the exception of portfolios sorted on the basis of book-to-market, statistically significant bias attributable to noisy prices is observed in non-January months as well, as t-statistics for the $EW - RW$ hedge portfolio differential in range in absolute value from 10.75 (for trading volume) to 15.06 (for inverse share price).

These results indicate that it is particularly important to control for noise in prices when studying January return data. This insight is relevant to researchers who consider implementing the IEW method. As noted, the IEW method does not correct for bias in the first period after portfolio formation. Blume and Stambaugh (1983) form portfolios as of the end of each December, but implement their correction in daily data. While the effect of failing to correct returns for a single day is likely to be minuscule, researchers implementing the IEW method in monthly return data will likely want to form portfolios at a date other than the end of December. A practical approach might be to form equal-weighted portfolios at the end of each November, skip December, and study IEW returns during the following year.[22]

Finally, it is noteworthy that, with the exception of the market-to-book ratio, the *EW* hedge portfolio return differential is insignificant in non-January months, indicating that the data does not support the existence of reliable return premia outside January for firm size, share price, trading volume, or illiquidy, even without correction for biases. At the same time, the difference in *EW* vs. *RW* hedge portfolio returns remains significant, indicating that the non-January mean returns are biased, even when they are statistically indistinguishable from zero.

B. Fama-MacBeth Regression and Subperiod Results

We next report on Table VI the results of estimating the return premia associated with the five firm-specific characteristics by means of Fama-MacBeth regressions of observed returns on each of the characteristics in turn, while controlling for risk as measured by market *Beta*. We report these results because such cross-sectional regressions are widely used in the empirical literature. Further, inferences supported by the cross-sectional regressions potentially differ from those obtained when comparing portfolio mean returns, both because of imposition of a specific functional form, and because the analysis is conducted at the level of individual securities rather than portfolios.[23]

<Table VI about here>

Since researchers most often estimate cross-sectional regressions by OLS, thereby placing equal weight on each observation, we limit this analysis to OLS estimation and *RW* estimation, where we estimate the regression by weighted-least-squares, using the prior-period gross return as the weighting variable. As noted earlier, weighting by prior-period gross returns corrects for the biases introduced by noisy prices, while continuing to give essentially the same weight to the information contained in large vs. small firm returns. Each cross-sectional regression is estimated on a monthly basis by OLS and RW. We record each estimate, and also the difference between the two estimates. The final coefficient estimate is the time-series means of the monthly estimates. The associated t-statistic is adjusted for autocorrelation in the monthly estimates as in Cooper, Gulen, and Schill (2008).[24]

We report empirical results for the full (1966 to 2009) sample, and for three subsamples comprising 1966–1982, 1983–2000, and 2001–2009. The first subperiod is comprised of NYSE-AMEX stocks, while Nasdaq stocks enter for the second subperiod. The final subperiod is mainly comprised of data following the 2001 introduction of decimal pricing, which led to substantial reductions in bid-ask spreads. Comparisons across subperiods allow evaluation of whether asset pricing anomalies have survived their initial discovery. Further, results for the final subperiod allow evaluation of whether biases due to noisy prices remain relevant after the 2001 decimalization of the U.S. stock markets.

The full-sample cross-sectional regressions support conclusions similar to those obtained on the basis of portfolio return comparisons and the existing literature. In particular, OLS regression estimates support the existence of return premia related to firm size (OLS t-statistic = -3.80), book-to market ratio (OLS t-statistic = 5.58), inverse share price (OLS t-statistic = -2.84), dollar trading volume (OLS t-statistic = -1.80 for NYSE stocks and -2.90 for Nasdaq stocks), and illiquidity (OLS t-statistic = 4.02 for NYSE stocks and 4.38 for Nasdaq stocks).

With regard to the central issue addressed in this paper, biases in estimated return premia attributable to noisy prices, the evidence on Table VI indicates that the biases are strong and pervasive when estimating return premia by means of cross-sectional OLS Fama-MacBeth regressions. Full sample t-statistics for the difference between the OLS and RW estimates range in absolute value from 2.30 for the book-to-market ratio to 13.33 for inverse share price. And, with only two exceptions, the t-statistic for the bias (difference between OLS and *RW* estimates) is statistically significant for each explanatory variable in each subperiod.[25]

We note that the estimated absolute magnitude of noise-induced biases has not uniformly decreased across subperiods. Focusing on firm size, for example, point estimates of the bias are -0.040, -0.081, and -0.057 for the three subperiods, and each is statistically significant. That the bias remains significant in the post-decimalization period provides indirect but strong evidence that the noise in security prices is attributable to sources in addition to bid-ask spreads, such as temporary price pressure attributable to accumulated order imbalances.

Horowitz, Loughran, and Savin (2000) document that the empirical relevance of firm size has diminished substantially in the years since papers describing the empirical size effect were first published. Consistent with their findings, we observe that the slope coefficient on firm size estimated by OLS decreases in absolute value from -0.243 during the 1966–1982 subperiod to -0.105 in the 1983 to 2000 subperiod. However, the estimated OLS coefficient for the most recent subperiod, 2001 to 2009, has again increased in absolute magnitude, to -0.238. Both the OLS and *RW* coefficient estimates for the size effect are statistically significant in the most recent period, and each is similar in magnitude to the corresponding estimate from the 1966–1982 subperiod. Hence, we conclude that reports of the demise of the size effect in returns may be premature.

Finally, we note that even though the return premium associated with *Beta* is statistically insignificant for all regression specifications in Table VI, we detect significant bias in the estimated return premium associated with beta. The mean differences between OLS and *RW* estimates of the beta premium reported in column "*DIF*" are uniformly positive and statistically different from zero, with only the exceptions in the final subperiod for results in Panel B and Panel C. The ability to detect a statistically significant bias in the $OLS - RW$ premium differential even while both the OLS and *RW* estimates are insignificant reflects that there is relatively little time-series variation in the monthly estimates of the bias.

C. Multivariate Fama-MacBeth Regressions

A key advantage of the univariate analyses reported in the preceding sections is that it is possible to form reasonably strong conjectures as to the likely sign of the cross-sectional correlation, and hence the direction of noise-induced bias, between levels of unobservable noise and individual explanatory variables. However, empirical asset-pricing studies using the Fama-MacBeth framework typically include several explanatory variables. Asparouhova, Bessembinder, and Kalcheva (2010) show that the direction of the bias in the individual OLS slope coefficients estimated in multivariate return regressions depend on the *partial* correlations between the variance of noise and the regression explanatory variables. Such partial correlations will depend on the combination of explanatory variables included in the multiple regression, and will likely be quite difficult to anticipate *a priori*.

Table VII reports results obtained in multivariate Fama-MacBeth regressions of monthly returns on various combinations of the explanatory variables. In general, conclusions as to which explanatory variables are reliably associated with returns after correcting for the effects of noise are sensitive to the set of explanatory variables included in the regression. Conclusions as to the direction of the bias in regression slope coefficients attributable to noise are similarly sensitive. Such sensitivity is to be expected given that a number of the explanatory variables are significantly correlated with each other.

<Table VII about here>

While the univariate evidence indicates that noise in prices is associated with significant bias for all explanatory variables examined here, the mean difference between the OLS and *RW* estimates in the multivariate specifications is not always significant. With the full set of explanatory variables included (specification (6)), we detect significant noise-induced bias in OLS coefficients on inverse share price, book-to-market ratio and market beta, but not on firm size, trading volume or illiquidity. Further, conclusions as to which explanatory variables are significantly affected by the correction for biases differs depending on the set of explanatory variables included in the regression. For example, the bias in the estimated coefficient on firm size is highly significant in specifications (1), (2), (3), and (5) but not in specifications (4) and (6). Further, the "*DIF*" coefficient for firm size is positive in specification (2) and negative in specifications (1), (3) and (5). The main implication of this mixed pattern of significance is that the likely effect of adjusting OLS coefficient estimates obtained in multivariate return regressions for biases attributable to noisy prices will be very difficult to ascertain *a priori*, and will typically need to be assessed empirically.

Finally, and perhaps most importantly, we note that inference as to whether particular explanatory variables have a significant effect on mean stock returns is altered by the correction for noise, in some, but not all, specifications. For example, in specification (5) the negative coefficient on firm size is statistically significant when estimated by OLS, but the coefficient is overstated by 40% relative to the corresponding bias-corrected (*RW*) estimate, which is not significant (t-statistic = -1.28). Here too, it would be very difficult to anticipate which coefficient estimates will potentially be rendered significant or insignificant by the correction for noise-induced bias. In a nutshell, the effect of correcting for noise in prices can be substantial, can alter statistical inference, and must be assessed empirically.

V. Conclusion

Researchers seek to understand the determinants of variation in mean returns across assets. Most empirical studies either compare returns across portfolios constructed by sorting on attributes of interest, or estimate regressions of returns on attributes or risk factors. However, if security prices contain noise, then return premium estimates obtained by comparison of equal-weighted mean returns across portfolios or by OLS return regressions are biased estimates of the true return differentials. The bias is relevant, because mean true returns, not mean observed returns, determine the rate of growth across time in expected prices and shareholder value.

This paper has two main goals. The first is to assess, by theory and simulation, the properties of a set of possible corrections for noisy security prices, under broader assumptions than allowed for in previous papers, including the possibility that the noise in prices may be serially correlated and/or contain a common component across stocks. The second is to provide illustrative examples of the potential importance of biases in estimated return premia, by comparing unadjusted (equal-weighted portfolio returns and OLS return regression parameter estimates) to corresponding estimates that are adjusted to mitigate the effects of noisy prices.

With regard to the first goal, we assess the properties of several return-weighting methods, including equal-weighting (EW), prior-gross-return weighting (RW), initial-equal-weighting (IEW), prior-firm-value weighting (VW), and annual-value weighting (AVW). We demonstrate that EW estimates are always biased in the presence of noisy prices. When the noise in prices is autocorrelated and/or contains a common component across stocks, the alternative methods may also be biased, but generally will be less so than the EW estimates. For plausible parameter estimates the remaining bias in RW or VW estimates is minimal.

Our analysis gives little reason to prefer RW over VW, or vice versa. However, the former provides a bias-corrected estimate that places equal weight on the information contained in each security, while the latter corrects for bias while weighting large firms more heavily. A researcher's choice between RW and VW methods may therefore depend on the desired weight to be given to the information contained in small versus large capitalization securities.

The analysis also indicates that the RW method performs slightly better than the IEW method, particularly when estimating cross-sectional mean returns, and that the VW method dominates the AVW method. The former result reflects in part that the IEW method does not correct for the effects of noisy prices in the first period after portfolios are formed, while the latter reflects that the AVW method corrects for the effects of noisy prices only in the first period after portfolios are formed.

With regard to the second goal, comparisons of returns across attribute-sorted decile portfolios as well as univariate Fama-MacBeth regressions reveal statistically significant biases in estimated return premia associated with every attribute considered, including firm size, market-to-book ratio, trading volume, share price, and illiquidity. However, the economic magnitude of the bias varies considerably, and is minimal in the case of the market-to-book ratio. In contrast, the bias attributable to noisy prices in return premia estimates associated with firm size, share price, trading volume and illiquidity can be substantial, equal to 50% or more of the corrected estimate.

The findings reported here indicate that correcting for the effects of noise in prices has significant effects on return premia estimates obtained from monthly return data. For the corrections to have substantial effects, the variance of the noise in prices must be substantial. Our findings therefore provide indirect support for the Hendershott, Li, Menkveld, and Seasholes (2011) finding that order imbalances lead to substantial noise in prices, and to transitory volatility in returns measured at the monthly horizon. One possibility is that the month-end prices used to compute calendar-month returns may contain more noise than other days of the month. Such a phenomenon could arise, for example, from trading used to move month-end prices strategically, along the lines documented by Carhart, Kaniel, Musto, and Reed (2002).

The empirical analysis presented here focused on monthly returns, and on five selected firm characteristics. Significant biases may well arise in other empirical applications. Any explanatory variable that is cross-sectionally correlated with the variance of the noise in prices is likely to be susceptible to bias in estimates of associated return premia. Also, the biases attributable to noisy prices will likely be more important in studies that consider returns measured over horizons shorter than the one-month interval considered here. We leave the assessment of biases obtained with alternative explanatory variables and shorter return horizons to future research.

Our analyses allow for non-zero correlation between the variance of noise and firm attributes and for possible dependence in noise realizations across time and securities. However, like Brennan and Wang (2010) we rely on the simplifying assumption that individual noise realizations in period t are independent of the random components of true returns in the same period. This assumption could be violated in some circumstances, e.g., if investors systematically over or under-react to contemporaneous firm-specific information arrivals. Assessing the effects of relaxing these assumptions on estimates of parameters of the noise distribution and on the effectiveness of the corrections considered here comprises a potentially interesting direction for future research.

Appendix A: Time-Series Implementation

We first assess the effect of estimation by the various weighting methods in the simplest scenario, namely, when estimating the mean and the regression coefficients for a single firm in a time-series setting. For this analysis, we drop the subscript n from the return equation to get $R_t = 1 + \alpha + \mathbf{X_t}\boldsymbol{\beta} + \epsilon_t = \tilde{\mathbf{X}}_\mathbf{t}\tilde{\boldsymbol{\beta}} + \epsilon_t$. Also, with some abuse of notation, let μ denote the time-series mean of R_t. The observed returns are $R_t^0 = R_t \frac{1+\delta_t}{1+\delta_{t-1}} = R_t D_t$, where $\delta_t = \sigma\delta_t^0$ and $\delta_t^0 = \rho\delta_{t-1}^0 + \sqrt{1-\rho^2}\xi_t$, with ξ_t being a zero-mean unit-variance i.i.d. random variable. Also, $E(\xi_t^3) = 0$. We use $D_{t,s}$ to denote $\frac{1+\delta_t}{1+\delta_{t-s}}$.

The following Lemma will be used to prove the propositions concerning the time-series properties of the proposed estimators.

LEMMA A1:

1. $E(\delta_t|\mathbf{X}) = 0$,
2. $E(D_t|\mathbf{X}) \approx 1 + \sigma^2(1-\rho)$,
3. $E(D_{t,s}|\mathbf{X}) \approx 1 + \sigma^2 - \sigma^2\rho^s$.

Proof: See the Internet Appendix.

Mean Estimates

PROPOSITION A1:

- $plim_{T\to\infty}\ \mu_{EW} \approx \mu\left(1 + \sigma^2(1-\rho)\right)$,
- $plim_{T\to\infty}\ \mu_{RW} \approx \frac{E(R_{t-1}R_t)}{E(R_{t-1})}\left(1 + \sigma^2(1-\rho)\rho\right)$,
- $plim_{T\to\infty}\ \mu_{RW(s)} \approx \frac{E(R_{t-1,s}R_t)}{E(R_{t-1,s})}\left(1 + \sigma^2(1-\rho)\rho^s\right)$,
- $plim_{T\to\infty}\ \mu_{VW} = \frac{E(P_tS)}{E(P_{t-1}S)} = \frac{E(P_t)}{E(P_{t-1})}$.

Proof: Immediate consequence from Lemma A1.

When estimating the means, the comparison between methods depends on the time-series properties of the prices (and therefore returns). If returns are independent in time, which would be the case if prices follow a martingale process, then both RW and $RW(s)$ provide estimates closer to the true mean returns than OLS does, with $RW(s)$'s bias in the limit being smaller than that of RW. The VW estimator is consistent under the restriction of prices following martingale (see footnote 6 as well). Generally, the VW bias would depend on $Cov(R_t, P_{t-1})$. The comparison between the magnitudes of the VW and the $RW(s)$ biases in the general case would depend on the time-series properties of the returns.

Regression Estimates

PROPOSITION A2:

- $plim_{T\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW}} \approx \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)\tilde{\boldsymbol{\beta}}$,
- $plim_{T\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW}} \approx \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)\rho\tilde{\boldsymbol{\beta}}$,
- $plim_{T\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s)}} \approx \tilde{\boldsymbol{\beta}} + \sigma^2(1-\rho)\rho^s\tilde{\boldsymbol{\beta}}$,
- $plim_{T\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW}} = \tilde{\boldsymbol{\beta}}$.

Proof: Immediate consequence from Lemma A1.

The VW weighting scheme provides consistent parameter estimates. Also, it is easy to see that the magnitude of the bias (in the limit) is the largest with OLS estimation, followed by the lagged return weighting method (RW), and then by the s-period lagged return scheme ($RW(s)$).

Appendix B: Cross-sectional Implementation

Using the notation and definitions introduced in subsection B of section II, in addition to introducing $D_{nt,s}$ to denote $\frac{1+\delta_{nt}}{1+\delta_{nt-s}}$, we can write the expressions for the probability limit of each (time t) cross-sectional estimator as follows.

A. Ordinary Least Squares, EW: $w_{nt} = \frac{1}{N}$

- $plim_{N\to\infty}\ \mu_{EW,t} = E(R^0_{nt}) = E(R_{nt}D_{nt}) = E(R_{nt}E(D_{nt}|\mathbf{X}, n))$, and
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}} =$
  $= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}R_{nt}D_{nt}\right) = \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}E(D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}.$

B. Weighting by the prior period's (gross) return, RW: $w_{nt} = R^0_{nt-1}$

- $plim_{N\to\infty}\ \mu_{RW,t} = \frac{E(R_{nt-1}R_{nt}D_{nt}D_{nt-1})}{E(R_{nt-1}D_{nt-1})} = \frac{E(R_{nt-1}R_{nt}E(D_{nt}D_{nt-1}|\mathbf{X},n))}{E\left(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}E(D_{nt-1}|\mathbf{X},n)\right)}.$
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}} = \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R^0_{nt-1}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}R^0_{nt-1}R^0_{nt}\right) =$
  $= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}|\mathbf{X}, n)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}.$

C. Weighting by the prior s periods' cumulative (gross) return, RW(s):

$w_{nt} = R^0_{nt-1,s} = R^0_{t-1}R^0_{t-2}...R^0_{t-1-s}.$

As $D_{nt,s} = \frac{1+\delta_{nt}}{1+\delta_{nt-s}}$ (thus, $D_{nt,1} = D_{nt}$), then $R^0_{nt-1,s} = R_{nt-1,s}D_{nt-1,s}$ and

- $plim_{N\to\infty}\ \mu_{RW(s),t} = \frac{E(R_{nt-1,s}R_{nt}D_{nt}D_{nt-1,s})}{E(R_{nt-1,s}D_{nt-1,s})} = \frac{E(R_{nt-1,s}R_{nt}E(D_{nt,s+1}|\mathbf{X},n))}{E(R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n))}.$
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}} = \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}D_{nt-1,s})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}R_{nt-1,s}R_{nt}D_{nt-1,s}D_{nt}) =$
  $= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X}, n)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt,s+1}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}.$

D. Weighting by the prior period's firm value, VW: $w_{nt} = S_nP^0_{nt-1}$

- $plim_{N\to\infty}\ \mu_{VW,t} = \frac{E(P_{nt-1}R_{nt}(1+\delta_{nt})S_n)}{E(P_{nt-1}(1+\delta_{nt-1})S_n)} = \frac{E(P_{nt}E(S_n(1+\delta_{nt})|\mathbf{X},n))}{E(P_{nt-1}E(S_n(1+\delta_{nt-1})|\mathbf{X},n))}.$
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}} =$
  $= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}(1+\delta_{nt-1})S_n\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}P_{nt-1}(1+\delta_{nt-1})S_nR_{nt}D_{nt}\right) =$
  $= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt-1}|\mathbf{X}, n)S_n\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt}|\mathbf{X}, n)S_n\right)\tilde{\boldsymbol{\beta}}.$

When $c = 0$ the following expressions, organized in a Lemma (with a proof provided in the Internet Appendix), can be easily derived (using second-order Taylor approximations):

LEMMA B1:

1. $E(\delta_{nt}|\mathbf{X}, n) = 0$,

2. $E\left(D_{nt}|\mathbf{X}, n\right) \approx 1 + \sigma_n^2(1-\rho)$,

3. $E(D_{nt}D_{nt-1}|\mathbf{X}, n) \approx 1 + \sigma_n^2(1-\rho^2)$,

4. $E(D_{nt,s}|\mathbf{X}, n) \approx 1 + \sigma_n^2 - \sigma_n^2\rho^s$.

Proof: See the Internet Appendix.

Proof of Proposition 1:

$$plim_{N\to\infty}\ \mu_{EW,t} = E\left(R_{nt}E(D_{nt}|\mathbf{X}, n)\right) \approx E\left(R_{nt}\left(1+\sigma_n^2(1-\rho)\right)\right).$$

$$plim_{N\to\infty}\ \mu_{RW,t} = \frac{E(R_{nt-1}R_{nt}E(D_{nt}D_{nt-1}|\mathbf{X},n))}{E(R_{nt-1}E(D_{nt-1}|\mathbf{X},n))} \approx \frac{E\left(R_{nt-1}R_{nt}\left(1+\sigma_n^2(1-\rho^2)\right)\right)}{E(R_{nt-1}(1+\sigma_n^2(1-\rho)))}.$$

$$plim_{N\to\infty}\ \mu_{RW(s),t} = \frac{E(R_{nt-1,s}R_{nt}E(D_{nt,s+1}|\mathbf{X},n))}{E(R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n))} \approx \frac{E\left(R_{nt-1,s}R_{nt}(1+\sigma_n^2-\sigma_n^2\rho^{s+1})\right)}{E(R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^s)}.$$

$$plim_{N\to\infty}\ \mu_{VW,t} = \frac{E(P_{nt}E(S_n(1+\delta_{nt})|\mathbf{X},n))}{E(P_{nt-1}E(S_n(1+\delta_{nt-1})|\mathbf{X},n))} = \frac{E(P_{nt}S_n)}{E(P_{nt-1}S_n)}.$$

Under the conditions of Proposition 1 the expressions in the propositions are a direct consequence of the expressions developed in Lemma B1.

Proof of Proposition 2:

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}} =$$
$$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}E(D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}} \approx$$
$$\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\left(1+\sigma_n^2(1-\rho)\right)\right)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}} =$$
$$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}|\mathbf{X}, n)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}} \approx$$
$$\left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}\left(1+\sigma_n^2(1-\rho)\right)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}\left(1+\sigma_n^2(1-\rho^2)\right)\right)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}} =$$
$$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n)\right)\right]^{-1}E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt,s+1}|\mathbf{X},n)\right)\tilde{\boldsymbol{\beta}} \approx$$
$$\left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^s)\right)\right]^{-1}E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^{s+1})\right)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}} =$$
$$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt-1}|\mathbf{X},n)S_n)\right]^{-1}E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt}|\mathbf{X},n)S_n)\tilde{\boldsymbol{\beta}} =$$
$$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}S_n)\right]^{-1}E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}S_n)\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}.$$

Under the conditions of Proposition 2 the expressions in the propositions are a direct consequence of the expressions above and those developed in Lemma B1.

The following expressions, organized in a Lemma, will be used to proof Propositions 3 and 4.

LEMMA B2:

1. $\bar{E}\left(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})}\right) = 1+\sigma^2c(1-\rho)$,

2. $\bar{E}(E(D_{nt})) \approx \bar{E}(E(1+\delta_{nt}-\delta_{nt-1}-\delta_{nt}\delta_{nt-1}+\delta_{nt-1}^2+\delta_{nt}\delta_{nt-1}^2)) = 1+\sigma^2(1-\rho)$,

3. $\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = 1+\sigma^2(1-\rho)(\rho-c\rho+c)$,

4. $\bar{E}(\frac{E(D_{nt-s}D_{nt})}{E(D_{nt-1})}) = 1+\sigma^2(1-\rho)\rho^{s-1}+\sigma^2c(1-2\rho^{s-1}+\rho^s)$.

Proof: See the Internet Appendix.

Proof of Proposition 3:

Substitute the expressions from Lemma B2 into the expressions below to get the expressions in the proposition.

$\bar{E}(plim_{N\to\infty}\ \mu_{EW,t}) = \bar{E}(E(R^0_{nt})) = \bar{E}(E(R_{nt}D_{nt})) = \bar{E}(E(R_{nt}E(D_{nt}|\mathbf{X},n)))$.

$\bar{E}(plim_{N\to\infty}\ \mu_{RW,t}) = \bar{E}(\frac{E(R_{nt-1}R_{nt}D_{nt}D_{nt-1})}{E(R_{nt-1}D_{nt-1})}) = \frac{E(R_{nt-1}R_{nt})}{E(R_{nt-1})}\bar{E}\frac{E(D_{nt}D_{nt-1})}{E(D_{nt-1})}$.

$\bar{E}(plim_{N\to\infty}\ \mu_{RW(s),t}) = \bar{E}(\frac{E(R_{nt-1,s}R_{nt}D_{nt}D_{nt-1,s})}{E(R_{nt-1,s}D_{nt-1,s})}) = \frac{E(R_{nt-1,s}R_{nt})}{E(R_{nt-1,s})}\bar{E}\frac{E(D_{nt}D_{nt-1,s})}{E(D_{nt-1,s})}$.

$\bar{E}(plim_{N\to\infty}\ \mu_{VW},t) = \bar{E}(\frac{E(P_{nt-1}R_{nt}(1+\delta_{nt})S_n)}{E(P_{nt-1}(1+\delta_{nt-1})S_n)}) = \frac{E(P_{nt-1}R_{nt}S_n)}{E(P_{nt-1}S_n)}\bar{E}(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})})$.

Proof of Proposition 4:

$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}}) =$

$= \bar{E}(\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}} D_{nt})\tilde{\boldsymbol{\beta}}) = \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} \bar{E}(E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}} D_{nt})\tilde{\boldsymbol{\beta}})) =$

$\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}} \bar{E}(D_{nt})\tilde{\boldsymbol{\beta}})) = \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(1+\sigma_n^2(1-\rho))\tilde{\boldsymbol{\beta}}))$.

$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}}) =$

$\bar{E}\{\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}})D_{nt-1})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}})D_{nt-1}D_{nt})\tilde{\boldsymbol{\beta}}\} =$

$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}))\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}))\tilde{\boldsymbol{\beta}}\}\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})})$.

$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}}) \approx \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(D_{nt-1,s}D_{nt})}{E(D_{nt-1,s})})$.

$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}}) \approx \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})})$.

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Footnotes

[1]See, for example, Aït-Sahalia, Mykland, Zhang (2005), Bandi and Russell (2006), Engle and Sun (2007), and Andersen, Bollerslev, and Meddahi (2011).

[2]While equal-weighting is likely the most common example of a weighting method leading to biased mean portfolio returns, other weighting methods, including fundamental weights (based on cash flows, earnings, dividends, etc.) also give biased portfolio means, as discussed further in Section II.

[3]The AVW method does reduce bias considerably as compared to equal-weighting, but the effect is primarily attributable to placing greater weight on large-capitalization firms that tend to have less noisy prices.

[4]Another possibility is the existence of mispricing that includes both permanent and temporary components. In this case the issues we address would still apply, and the corrections proposed would still be effective, for the effects on return premia estimates of temporary deviations around the "true value + permanent mispricing" benchmark.

[5]Notably, there would be no upward bias in the average log return. However, see Ferson and Korajczyk (1995), who articulate several reasons that it may not be appropriate to use continuously compounded returns when testing discrete-time asset pricing models. In any case, the large majority of empirical analyses focus on simple rather than log returns.

[6]In general, $E(R_t) = \frac{E(P_t)}{E(P_{t-1})} - \frac{cov(R_t, P_{t-1})}{E(P_{t-1})}$. Setting the covariance to zero gives the result.

[7]Note the distinction between this observation and the well known fact that the arithmetic mean return exceeds the geometric mean return, unless the variance of returns is zero. In contrast, the expected observed return exceeds the expected true return *only*

if prices contain noise, i.e. temporary deviations of price from underlying value.

[8]Fisher and Weaver (1992) independently develop a method for correcting returns to equal-weighted stock indices for noisy prices. Their method focuses on the ratio of two-period to one-period index returns, but is equivalent to weighting by prior-period gross returns.

[9]In particular, some elements of $\mathbf{X_{nt}}$ can be identical across n, allowing for commonality in returns.

[10]In a sample where both N and T are assumed large, sequential consistency is consistency established when first N goes to infinity and only then T does.

[11]Note that the sequential consistency applied here would also apply in the previous subsection for the case of $c = 0$.

[12]Note that this implies a cross-sectional standard deviation for σ_n of 0.035. This is less than the corresponding estimate reported by Brennan and Wang, which is 0.056. Clearly the estimated distribution of sigma is right-skewed. By not accommodating this skewness we are being conservative — accommodating the right skewness would increase the bias in unadjusted estimates. See equation (4) in Asparouhova, Bessembinder, and Kalcheva (2010).

[13]This estimate is also based on results from Brennan and Wang. They report an (adjusted) R-squared of 0.055, implying a correlation of ± 0.235, in a cross-sectional regressions of σ_n on firm characteristics. While the authors use an array of explanatory variables, for simplicity we load the correlation on firm value only. Brennan and Wang also report that empirical estimates of ρ are negatively related to firm size. However, the R-squared is only 0.01. We assessed the effect of accommodating a corresponding negative correlation between ρ and firm size in the simulations, and found results to be

wholly unaffected.

[14]Cross-sectional variation in expected returns is at odds with this assumption, since such variation implies that securities with returns higher than the cross-sectional mean in period $t-1$ tend to also have high returns in period t. Conrad and Kaul (1998) also observe that the cross-sectional covariance between current and lagged returns depends on cross-sectional variation in mean returns, and demonstrate that a significant proportion of observed "momentum" profits are attributable to variation in unconditional mean returns.

[15]Note that in a CAPM framework the cross-sectional standard deviation of expected return is the average market return times the cross-sectonal standard deviation of beta. Given an expected market return of 1%, a cross-sectional standard deviation of beta equal to 1.0 would be required to induce a cross-sectional standard deviation of expected returns as large as 1%.

[16]This reflects that the cross-sectional covariance $Cov(R_{nt}, R_{nt-s})$ grows larger with s when there is cross-sectional variation in mean returns, and that the IEW method weights by longer horizon returns as compared to the RW method.

[17]There is some evidence concerning the degree of commonality in measures of illiquidity. Chordia, Roll, and Subrahmanyam (2000) report adjusted R-squared statistics for cross-sectional regressions of firm-level on marketwide illiquidity measures that are uniformly less than two percent. Similarly, Hasbrouck and Seppi (2001) report that the first principal component explains less than eight percent of the variation in signed order flow across stocks. While these estimates indicate that the degree of commonality in particular contributors to noisy prices is not high, they do not comprise direct evidence on commonality in noise.

[18]The book-to-market ratio is defined as the sum of fiscal year-end book equity (Compustat item #60) and balance sheet deferred taxes (Compustat item #74), divided

by the CRSP market capitalization in December of the corresponding year. As in Fama and French (1992), the value of BM for July of year t to June of year $t+1$ was computed using accounting data at the end of year $t-1$, and book-to-market ratio values greater than the 0.995 fractile or less than the 0.005 fractile were set equal to the 0.995 and 0.005 fractile values, respectively. The book value of common equity (Compustat data 60) is not generally available prior to 1962, see Fama and French(1992), p.429.

[19]Given that the interpretation of trading volume potentially differs across markets, we use in the regression-based analyses indicator variables to allow for separate slope coefficients on trading volume (and the illiquidity measure) for NYSE/AMEX and Nasdaq-listed stocks.

[20]*Beta* is estimated every December for all stocks with at least 24 return observations over the prior 60 months, with the qualification that since the factor estimation begins in July 1963, the factor loadings in the first month of the regression period (January 1966) were estimated from 30 observations. The Dimson (1979) procedure with one lag is implemented to allow for potential thin trading.

[21]The lone exception is the differential between the EW mean and the IEW mean book-to-market premium, for which the t-statistic is 1.34.

[22]As noted, the IEW results reported on Tables II and III of this paper are based on portfolios formed at the end of each July.

[23]Ang, Liu, and Schwarz (2010) show theoretically and empirically that individual-stock regressions have better large-sample statistical properties than portfolio-based regressions.

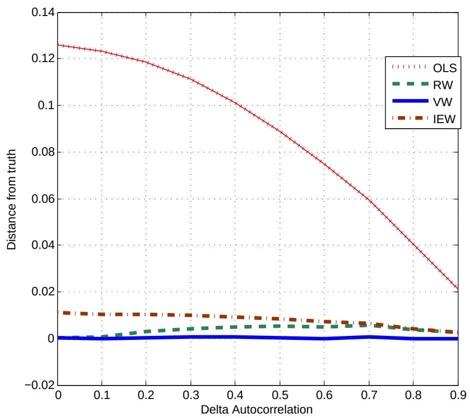
[24]Autocorrelations in monthly return premia estimates are modest. Across the seven full sample return premium estimates on Table VI (including separate Nasdaq and NYSE coefficients for trading volume and illiquidity), the average first-order autocorrelation in

the OLS return premium estimates is 0.076, while the first-order autocorrelation in the RW return premium estimates averages 0.083.

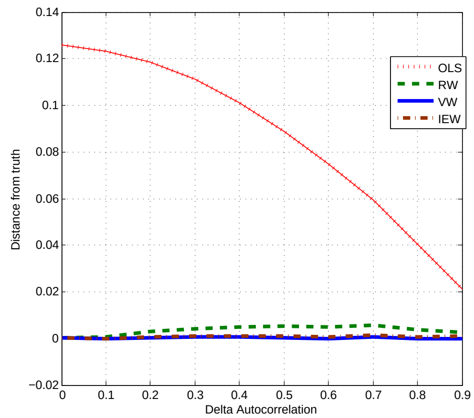
[25]The two exceptions are for the book-to-market ratio in the 2001–2009 period and dollar volume for NYSE stocks during the 1966–1982 period.

Figure 1. Simulation Results: Distance of the Estimated Slope Coefficient on Illiquidity from the True Value of 0.15

Panel A: Estimators line plots when $c = 0$

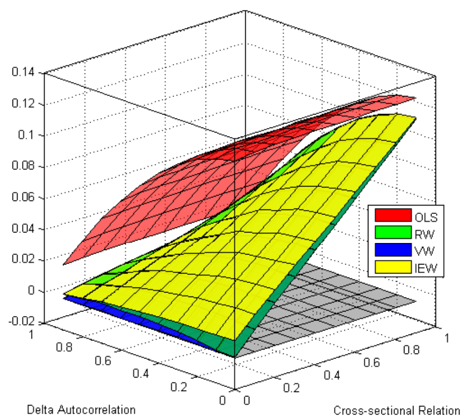


(a) IEW using t=1 to 12.

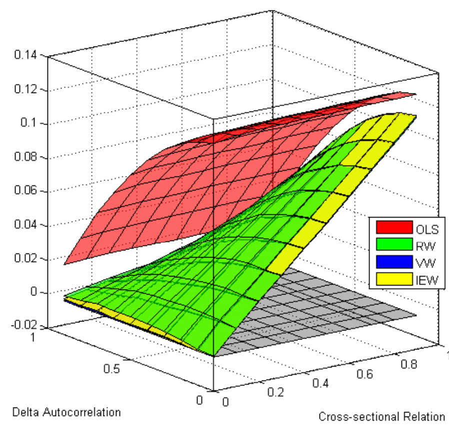


(b) IEW using t=2 to 12.

Panel B: Estimators surface plots

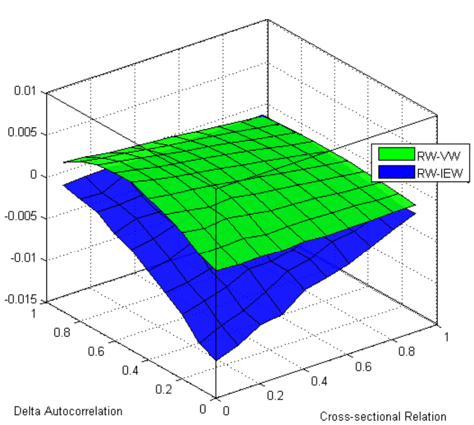


(c) IEW using t=1 to 12.

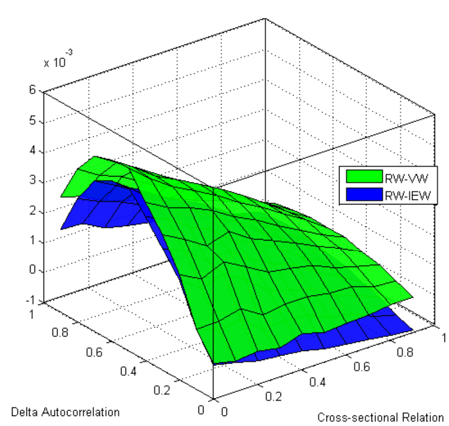


(d) IEW using t=2 to 12.

Panel C: Differences of VW and IEW from RW.



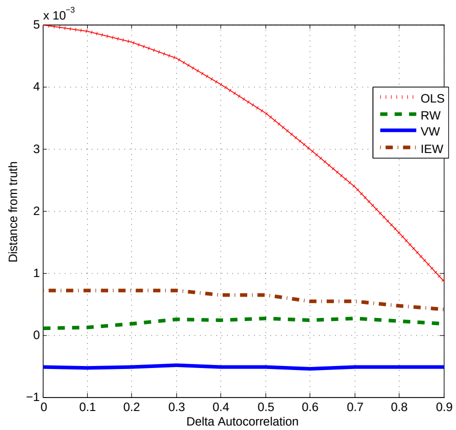
(e) IEW using t=1 to 12.



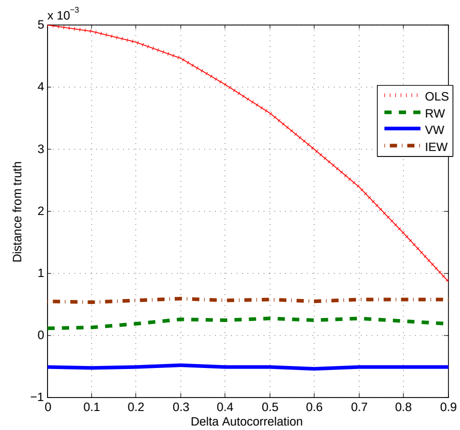
(f) IEW using t=2 to 12.

Figure 2. Simulation Results: Distance of the Estimated Cross-sectional Mean Return from the True Mean of 0.01

Panel A: Estimators line plots when $c = 0$

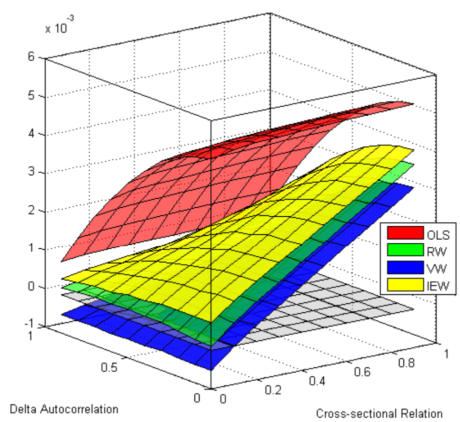


(a) IEW using t=1 to 12.

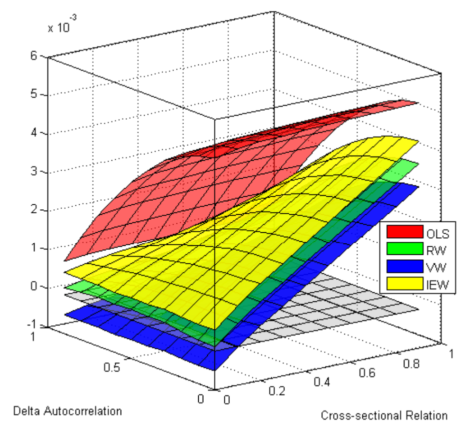


(b) IEW using t=2 to 12.

Panel B: Estimators surface plots

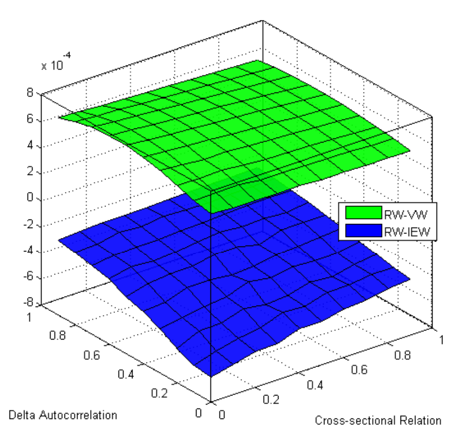


(c) IEW using t=1 to 12.

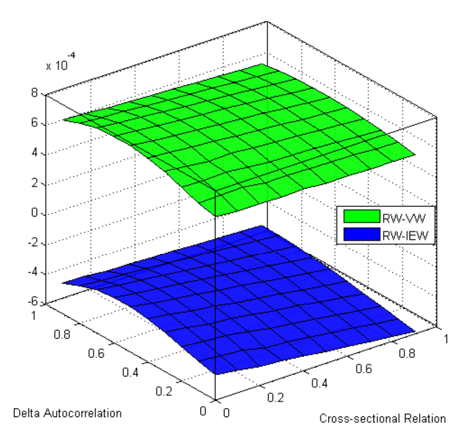


(d) IEW using t=2 to 12.

Panel C: Differences of VW and IEW from RW



(e) IEW using t=1 to 12.



(f) IEW using t=2 to 12.

Table I. Summary Statistics and Correlations

Panel A represents the time-series averages of monthly cross-sectional means for a sample that averages 3762 stocks, over 528 months from January 1966 to December 2009. Monthly returns are in excess of the treasury interest rate. Firm size is in billions. Book-to-market ratio (BM) is winsorized at the 0.005 and the 0.995 fractiles of the full sample by setting the outlying values to the 0.005 and the 0.995 fractiles respectively. Share price is in dollars. Volume is in $ millions per month. Volume for Nasdaq stocks is available after 1983. *Illiq* is the Amihud (2002) illiquidity measure. Panel B presents time-series of monthly cross-sectional correlations (*Illiq* and *Dvol* are standardized as per Eq. 3 and Eq. 4 in Amihud (2002)) between firm characteristics.

Panel A: Summary Statistics

Variable	Mean	Median	St.dev.
Return	0.827	0.976	6.066
Firm size	1.182	0.559	1.157
BM	0.931	0.836	0.378
Share price	25.551	22.217	10.305
Volume	142.299	32.284	223.229
Illiq	7.285	5.235	6.968

Panel B: Correlation Matrix of Transformed Firm Characteristics

Variable	*Return*	*Size*	*log(BM)*	*InvPrice*	*Dvol*
Return	1	–	–	–	–
Size	-0.010	1	–	–	–
log(BM)	0.029	-0.287	1	–	–
InvPrice	0.004	-0.783	0.213	1	–
Dvol	-0.017	0.886	-0.327	-0.691	1
Illiq	0.019	-0.323	0.148	0.353	-0.339

Table II. Mean Returns to Attribute-Sorted Portfolios, January 1966 to December 2009

The table reports time-series means of monthly returns to the extreme of the ten attribute-sorted portfolios and to the corresponding hedge portfolio. Portfolio returns for month *t* are measured on an equal-weighted (EW), return-weighted (RW, weight is period *t-1* gross return), equal-initial-weighted (IEW, weight is cumulative gross return from portfolio formation through month *t-1*, prior-month-value-weighted (VW, weight is month *t-1* market capitalization), and Annual-value-weighted (AVW, weight is previous December market capitalization) basis. Firms are assigned to portfolios based on attributes measured in July. T-statistics are reported in parentheses.

Extreme Deciles and Hedge Portfolio					**Hedge Portfolio Differential**									
	10	1	10-1	(T-stats)	Estimates					(T-stats)				
Panel A: Size														
EW	0.462	1.888	-1.425	(-4.43)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.448	1.410	-0.961	(-3.07)	*EW*	-0.463	-0.681	-0.908	-0.549	*EW*	(-13.52)	(-9.76)	(-9.04)	(-5.61)
IEW	0.450	1.194	-0.743	(-2.43)	*RW*	–	-0.218	-0.444	-0.085	*RW*	–	(-4.07)	(-5.14)	(-0.92)
VW	0.371	0.888	-0.517	(-1.71)	*IEW*	–	–	-0.226	0.132	*IEW*	–	–	(-3.37)	(1.55)
AVW	0.365	1.241	-0.876	(-2.86)	*VW*	–	–	–	0.358	*VW*	–	–	–	(5.40)
Panel B: Book-to-Market														
EW	1.517	0.148	1.369	(6.03)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	1.301	0.024	1.277	(5.69)	*EW*	0.092	0.070	0.552	0.637	*EW*	(2.21)	(1.34)	(2.32)	(2.68)
IEW	1.331	0.033	1.298	(5.94)	*RW*	–	-0.021	0.460	0.545	*RW*	–	(-0.42)	(2.05)	(2.44)
VW	1.031	0.214	0.816	(3.18)	*IEW*	–	–	0.481	0.566	*IEW*	–	–	(2.12)	(2.49)
AVW	0.962	0.230	0.731	(2.81)	*VW*	–	–	–	0.084	*VW*	–	–	–	(1.73)
Panel C: Inverse Price														
EW	1.832	0.579	1.252	(3.27)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	1.214	0.569	0.645	(1.74)	*EW*	0.607	0.800	1.088	0.768	*EW*	(15.04)	(10.14)	(5.94)	(2.86)
IEW	1.035	0.583	0.452	(1.27)	*RW*	–	0.193	0.480	0.161	*RW*	–	(3.16)	(2.72)	(0.60)
VW	0.570	0.405	0.164	(0.40)	*IEW*	–	–	0.287	-0.031	*IEW*	–	–	(1.69)	(-0.12)
AVW	0.872	0.388	0.483	(1.04)	*VW*	–	–	–	-0.319	*VW*	–	–	–	(-2.03)
Panel D: Volume														
EW	0.407	1.605	-1.198	(-4.52)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.396	1.243	-0.846	(-3.28)	*EW*	-0.352	-0.514	-0.839	-0.717	*EW*	(-12.36)	(-9.61)	(-6.51)	(-5.60)
IEW	0.414	1.098	-0.684	(-2.69)	*RW*	–	-0.162	-0.487	-0.365	*RW*	–	(-3.67)	(-3.98)	(-2.96)
VW	0.354	0.713	-0.359	(-1.70)	*IEW*	–	–	-0.324	-0.203	*IEW*	–	–	(-2.80)	(-1.71)
AVW	0.344	0.825	-0.481	(-2.28)	*VW*	–	–	–	0.121	*VW*	–	–	–	(4.09)
Panel E: Illiquidity Ratio														
EW	1.580	0.441	1.139	(3.95)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	1.211	0.430	0.780	(2.77)	*EW*	0.358	0.464	0.726	0.665	*EW*	(12.77)	(8.33)	(5.10)	(5.70)
IEW	1.108	0.433	0.675	(2.38)	*RW*	–	0.105	0.367	0.306	*RW*	–	(2.42)	(2.77)	(2.78)
VW	0.769	0.356	0.413	(1.42)	*IEW*	–	–	0.262	0.200	*IEW*	–	–	(2.26)	(1.99)
AVW	0.826	0.351	0.474	(1.74)	*VW*	–	–	–	-0.061	*VW*	–	–	–	(-0.98)

Table III. Mean Returns to Attribute-Sorted Portfolio, January 1966 to December 2009, with Price Filter

The table replicates Table II, except that stocks with price per share less than \$5 as of the end of previous month are excluded. The table reports time-series means of monthly returns to the extreme of the ten attribute-sorted portfolios and to the corresponding hedge portfolio. Portfolio returns for month t are measured on an equal-weighted (EW), return-weighted (RW, weight is period *t-1* gross return), equal-initial-weighted (IEW, weight is cumulative gross return from portfolio formation through month *t-1*, prior-month-value-weighted (VW, weight is month *t-1* market capitalization), and Annual-value-weighted (AVW, weight is previous December market capitalization) basis. Firms are assigned to portfolios based on attributes measured in July. T-statistics are reported in parentheses. An * denotes that the estimate in the table differs significantly (p-value < 0.05) from the corresponding estimate reported in Table II.

Extreme Deciles and Hedge Portfolio					**Hedge Portfolio Differential**									
	10	1	10-1	(T-stats)	Estimates					(T-stats)				
Panel A: Size														
EW	0.437	0.792*	-0.355*	(-1.97)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.424	0.716*	-0.291*	(-1.62)	*EW*	-0.064*	-0.017*	0.022*	-0.006*	*EW*	(-6.31)	(-0.65)	(0.40)	(-0.12)
IEW	0.428	0.766*	-0.338*	(-1.91)	*RW*	–	0.046*	0.086*	0.057	*RW*	–	(1.92)	(1.57)	(1.05)
VW	0.362	0.740	-0.378	(-2.02)	*IEW*	–	–	0.040*	0.010	*IEW*	–	–	(0.83)	(0.21)
AVW	0.352	0.701*	-0.349*	(-1.86)	*VW*	–	–	–	-0.029*	*VW*	–	–	–	(-1.24)
Panel B: Book-to-Market														
EW	0.892*	0.090	0.802*	(4.02)		*RW*	*IEW*	*VW*	*DVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.842*	0.042	0.800*	(3.99)	*EW*	0.002*	0.028	0.182*	0.224*	*EW*	(0.14)	(0.79)	(0.98)	(1.20)
IEW	0.883*	0.109	0.773*	(3.83)	*RW*	–	0.026	0.180*	0.222*	*RW*	–	(0.89)	(0.95)	(1.17)
VW	0.830	0.211	0.619	(2.65)	*IEW*	–	–	0.154*	0.196*	*IEW*	–	–	(0.80)	(1.01)
DVW	0.811	0.234	0.577	(2.48)	*VW*	–	–	–	0.042	*VW*	–	–	–	(1.33)
Panel C: Inverse Price														
EW	0.480*	0.565	-0.084*	(-0.45)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.410*	0.557	-0.147*	(-0.79)	*EW*	0.062*	0.015*	0.097*	0.084*	*EW*	(5.45)	(0.48)	(0.71)	(0.57)
IEW	0.469*	0.569	-0.100*	(-0.54)	*RW*	–	-0.046*	0.035*	0.022	*RW*	–	(-1.54)	(0.26)	(0.15)
VW	0.219	0.401	-0.182	(-0.75)	*IEW*	–	–	0.081	0.068	*IEW*	–	–	(0.61)	(0.47)
AVW	0.215	0.385	-0.169	(-0.68)	*VW*	–	–	–	-0.012*	*VW*	–	–	–	(-0.33)
Panel D: Volume														
EW	0.361*	0.800*	-0.439*	(-2.40)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.360	0.740*	-0.380*	(-2.09)	*EW*	-0.059*	-0.054*	-0.150*	-0.128*	*EW*	(-5.03)	(-1.65)	(-1.56)	(-1.32)
IEW	0.384	0.770	-0.385	(-2.17)	*RW*	–	0.005*	-0.091*	-0.068*	*RW*	–	(0.17)	(-0.96)	(-0.72)
VW	0.342	0.631	-0.288	(-1.78)	*IEW*	–	–	-0.096*	-0.073	*IEW*	–	–	(-1.03)	(-0.77)
AVW	0.325*	0.637	-0.311	(-1.91)	*VW*	–	–	–	0.022*	*VW*	–	–	–	(1.23)
Panel E: Illiquidity Ratio														
EW	0.892*	0.407	0.484*	(2.81)		*RW*	*IEW*	*VW*	*AVW*		*RW*	*IEW*	*VW*	*AVW*
RW	0.828*	0.399	0.429*	(2.50)	*EW*	0.054*	0.001*	0.067*	0.079*	*EW*	(4.97)	(0.04)	(0.76)	(0.91)
IEW	0.883	0.399	0.483	(2.88)	*RW*	–	-0.053*	0.012*	0.024*	*RW*	–	(-2.34)	(0.14)	(0.28)
VW	0.764	0.347	0.417	(2.47)	*IEW*	–	–	0.066*	0.078	*IEW*	–	–	(0.81)	(0.94)
AVW	0.743	0.338	0.405	(2.38)	*VW*	–	–	–	0.011	*VW*	–	–	–	(0.50)

Table IV. Alphas to Attribute-Sorted Portfolios, January 1966 to December 2009

The table reports alphas for the extreme attribute-sorted portfolios and the corresponding hedge portfolio, estimated as intercepts in time-series regressions of monthly portfolio returns on a four-factor asset pricing model. We also report alphas obtained when the dependent variable is the difference in differences: RW(10-1) less VW(10-1), RW(10-1) less VW(10-1) and EW(10-1) less VW(10-1). Portfolio returns for month t are measured on an equal-weighted (EW), return-weighted (RW, weight is period *t-1* gross return), and prior-month-value-weighted (VW, weight is month *t-1* market capitalization) basis. Firms are assigned to portfolios based on attributes measured in July. P(GRS) is the p-value of the F-statistic of Gibbons, Ross, and Shanken (1989), and pertains to the hypothesis that intercepts for all ten attribute sorted portfolios are simultaneously equal to zero. T-statistics are reported in parentheses.

	Extreme Deciles and Hedge Portfolio					**Hedge Portfolio Differential**					
	10	1	10-1	(T-stats)	p(GRS)	Estimates			(T-stats)		
Panel A: Size											
EW	0.102	1.228	-1.126	(-4.33)	0.00		*RW*	*VW*		*RW*	*VW*
RW	0.084	0.737	-0.653	(-2.66)	0.00	*EW*	-0.473	-1.011	*EW*	(-10.46)	(-9.40)
VW	0.040	0.156	-0.115	(-0.57)	0.22	*RW*	–	-0.538	*RW*	–	(-6.05)
Panel B: Book-to-Market											
EW	0.787	0.015	0.772	(3.84)	0.00		*RW*	*VW*		*RW*	*VW*
RW	0.553	-0.115	0.669	(3.58)	0.00	*EW*	0.102	0.793	*EW*	(1.98)	(3.05)
VW	0.146	0.168	-0.021	(-0.13)	0.85	*RW*	–	0.690	*RW*	–	(2.80)
Panel C: Inverse Price											
EW	1.201	0.169	1.031	(2.98)	0.00		*RW*	*VW*		*RW*	*VW*
RW	0.559	0.156	0.402	(1.25)	0.00	*EW*	0.628	1.234	*EW*	(11.10)	(5.27)
VW	-0.121	0.081	-0.203	(-0.56)	0.15	*RW*	–	0.606	*RW*	–	(2.74)
Panel D: Volume											
EW	0.058	0.955	-0.896	(-4.22)	0.00		*RW*	*VW*		*RW*	*VW*
RW	0.042	0.586	-0.543	(-2.64)	0.01	*EW*	-0.352	-0.850	*EW*	(-9.67)	(-6.21)
VW	0.030	0.076	-0.046	(-0.34)	0.48	*RW*	–	-0.497	*RW*	–	(-3.72)
Panel E: Illiquidity Ratio											
EW	0.805	0.112	0.692	(3.21)	0.00		*RW*	*VW*		*RW*	*VW*
RW	0.429	0.097	0.331	(1.60)	0.02	*EW*	0.361	0.898	*EW*	(10.27)	(7.28)
VW	-0.166	0.039	-0.205	(-1.11)	0.32	*RW*	–	0.537	*RW*	–	(4.59)

Table V. January vs. Non-January Returns, January 1966 to December 2009

The table reports time-series means of monthly returns to the extreme of the ten attribute-sorted portfolios and to the corresponding hedge portfolio for January month and separate for non-January month. Portfolio returns for month t are measured on an equal-weighted (EW), return-weighted (RW, weight is period *t-1* gross return), and prior-month-value-weighted (VW, weight is month *t-1* market capitalization) basis. Firms are assigned to portfolios based on attributes measured in July. T-statistics are reported in parentheses.

	Extreme Deciles and the Hedge Portfolio									Hedge Portfolio Differentials		
	EW_{10}	EW_1	EW_{10-1}	RW_{10}	RW_1	RW_{10-1}	VW_{10}	VW_1	VW_{10-1}	*EW-RW*	*RW-VW*	*EW-VW*
Panel A: Size												
January	0.708	13.764	-13.055	0.646	12.34	-11.693	0.520	10.237	-9.717	-1.361	-1.976	-3.338
(t-stats)	(0.86)	(7.92)	(-8.77)	(0.79)	(7.57)	(-8.38)	(0.65)	(6.89)	(-7.67)	(-7.32)	(-4.60)	(-6.07)
Non-January	0.440	0.819	-0.379	0.430	0.427	0.003	0.357	0.047	0.310	-0.383	-0.306	-0.689
(t-stats)	(2.00)	(2.41)	(-1.37)	(1.97)	(1.26)	(0.01)	(1.76)	(0.14)	(1.11)	(-12.39)	(-3.68)	(-7.54)
Panel B: Book-to-Market												
January	9.345	3.784	5.560	8.615	3.525	5.090	4.227	0.557	3.670	0.469	1.420	1.890
(t-stats)	(6.06)	(2.97)	(4.86)	(5.67)	(2.96)	(4.30)	(5.67)	(0.53)	(2.55)	(1.47)	(1.73)	(1.83)
Non-January	0.804	-0.182	0.987	0.634	-0.294	0.929	0.740	0.183	0.556	0.057	0.372	0.430
(t-stats)	(2.61)	(-0.55)	(4.56)	(2.06)	(-0.90)	(4.36)	(2.44)	(0.66)	(2.27)	(1.66)	(1.60)	(1.78)
Panel C: Inverse Price												
January	16.265	0.555	15.710	14.689	0.495	14.194	12.858	0.457	12.401	1.516	1.792	3.309
(t-stats)	(7.66)	(0.67)	(8.46)	(7.50)	(0.60)	(8.27)	(6.78)	(0.56)	(7.36)	(5.77)	(2.69)	(4.28)
Non-January	0.534	0.581	-0.047	0.003	0.576	-0.573	-0.535	0.400	-0.935	0.525	0.362	0.888
(t-stats)	(1.32)	(2.58)	(-0.15)	(0.007)	(2.57)	(-1.80)	(-1.19)	(1.96)	(-2.44)	(15.06)	(1.99)	(4.81)
Panel D: Volume												
January	0.987	11.295	-10.308	0.904	10.184	-9.279	0.542	5.964	-5.421	-1.028	-3.857	-4.886
(t-stats)	(1.05)	(7.07)	(-7.79)	(0.98)	(6.70)	(-7.37)	(0.66)	(4.60)	(-5.09)	(-7.88)	(-7.47)	(-8.79)
Non-January	0.355	0.734	-0.379	0.351	0.439	-0.088	0.337	0.241	0.095	-0.291	-0.183	-0.475
(t-stats)	(1.38)	(2.64)	(-1.66)	(1.38)	(1.58)	(-0.39)	(1.61)	(1.08)	(0.49)	(-10.75)	(-1.60)	(-4.03)
Panel E: Illiquidity Ratio												
January	11.855	1.104	10.750	10.717	1.042	9.675	8.006	0.563	7.442	1.075	2.232	3.307
(t-stats)	(6.91)	(1.23)	(7.84)	(6.59)	(1.18)	(7.43)	(5.28)	(0.70)	(5.78)	(7.18)	(5.36)	(6.75)
Non-January	0.656	0.381	0.275	0.356	0.375	-0.018	0.119	0.338	-0.219	0.294	0.200	0.494
(t-stats)	(2.09)	(1.60)	(1.08)	(1.13)	(1.59)	(-0.07)	(0.36)	(1.65)	(-0.79)	(11.53)	(1.46)	(3.43)

Table VI. Univariate Fama-MacBeth Regressions, January 1966 to December 2009

Reported are results of implementing cross-sectional Fama-MacBeth regressions of monthly stock returns, relying on NYSE-Amex stocks from 1966 to 2009 and including Nasdaq stocks from 1983 to 2009. Panels A through E report results for the different firm-specific characteristics. The coefficients reported in Column OLS are the time-series means of the monthly cross-sectional OLS regression estimates, while coefficients reported in Column *RW* are the time-series means of the monthly cross-sectional WLS regression estimates, where the weighting variable is one plus previous month return. The coefficients reported in Column DIF are the time-series means of the difference between the OLS and WLS coefficient. *T*-statistics are reported in parentheses and adjusted for autocorrelation as in footnote 13 in Cooper, Gulen, and Schill (2008).

		Period	OLS (*T*-stat.)	*RW* (*T*-stat.)	DIF (*T*-stat.)
Panel A	*Size*	*1966–2009*	-0.186 (-3.80)	-0.125 (-2.61)	-0.060 (-11.50)
		1966–1982	-0.243 (-2.75)	-0.203 (-2.33)	-0.040 (-7.55)
		1983–2000	-0.105 (-1.39)	-0.024 (-0.32)	-0.081 (-8.73)
		2001–2009	-0.238 (-2.53)	-0.180 (-2.03)	-0.057 (-4.20)
	Beta	*1966–2009*	0.015 (0.12)	-0.040 (-0.34)	0.055 (7.26)
		1966–1982	-0.076 (-0.40)	-0.112 (-0.60)	0.036 (5.05)
		1983–2000	-0.067 (-0.38)	-0.136 (-0.80)	0.068 (4.86)
		2001–2009	0.355 (1.01)	0.287 (0.84)	0.067 (2.92)
Panel B	*log(BM)*	*1966–2009*	0.463 (5.58)	0.446 (5.41)	0.017 (2.30)
		1966–1982	0.499 (3.24)	0.453 (3.01)	0.045 (5.15)
		1983–2000	0.413 (5.00)	0.412 (4.88)	0.001 (0.086)
		2001–2009	0.500 (2.01)	0.504 (2.00)	-0.003 (-0.14)
	Beta	*1966–2009*	0.132 (0.94)	0.051 (0.39)	0.081 (3.86)
		1966–1982	0.120 (0.53)	0.045 (0.20)	0.074 (6.69)
		1983–2000	-0.055(-0.33)	-0.116 (-0.71)	0.061 (5.43)
		2001–2009	0.558 (1.18)	0.420 (1.10)	0.137 (1.30)
Panel C	*InvPrice*	*1966–2009*	0.312 (2.84)	0.160 (1.49)	0.152 (13.33)
		1966–1982	0.385 (1.87)	0.273 (1.35)	0.112 (9.21)
		1983–2000	0.141 (0.88)	-0.053 (-0.33)	0.194 (10.68)
		2001–2009	0.516 (2.21)	0.373 (1.75)	0.143 (4.16)
	Beta	*1966–2009*	-0.007 (-0.06)	-0.037 (-0.33)	0.029 (4.47)
		1966–1982	-0.038 (-0.20)	-0.060 (-0.32)	0.022 (3.82)
		1983–2000	-0.058 (-0.36)	-0.101 (-0.64)	0.0427 (3.21)
		2001–2009	0.152 (0.52)	0.134 (0.46)	0.018 (1.08)
Panel D	*NYdvol*	*1966–2009*	-0.093 (-1.89)	-0.057 (-1.14)	-0.035 (-5.36)
		1966–1982	-0.022 (-0.55)	-0.024 (-0.64)	0.002 (0.56)
		1983–2000	-0.071 (-0.73)	-0.025 (-0.25)	-0.045 (-4.50)
		2001–2009	-0.269 (-1.89)	-0.181 (-1.30)	-0.088 (-4.08)
	NAdvol	*1983–2009*	-0.241 (-2.90)	-0.151 (-1.87)	-0.089 (-7.34)
		1983–2000	-0.132 (-1.79)	-0.054 (-0.76)	-0.078 (-6.59)
		2001–2009	-0.460 (-2.17)	-0.347 (-1.69)	-0.113 (-3.94)
	Beta	*1966–2009*	0.104 (0.81)	0.024 (0.19)	0.080 (9.04)
		1966–1982	0.038 (0.17)	-0.021 (-0.09)	0.059 (6.72)
		1983–2000	-0.017 (-0.10)	-0.112 (-0.67)	0.094 (6.45)
		2001–2009	0.474 (1.31)	0.384 (1.09)	0.090 (3.09)
Panel E	*NYilliq*	*1966–2009*	0.116 (4.02)	0.070 (2.74)	0.045 (6.38)
		1966–1982	0.110 (2.35)	0.078 (1.77)	0.031 (6.43)
		1983–2000	0.153 (2.86)	0.083 (1.81)	0.069 (4.32)
		2001–2009	0.055 (1.89)	0.030 (1.15)	0.024 (2.96)
	NAilliq	*1983–2009*	0.062 (4.38)	0.045 (3.20)	0.017 (6.77)
		1983–2000	0.060 (2.96)	0.042 (2.11)	0.017 (5.48)
		2001–2009	0.065 (2.92)	0.050 (2.38)	0.015 (4.22)
	Beta	*1966–2009*	0.071 (0.54)	0.001 (0.01)	0.070 (8.56)
		1966–1982	0.039 (0.18)	-0.019 (-0.09)	0.059 (6.22)
		1983–2000	-0.080 (-0.44)	-0.151 (-0.85)	0.071 (5.56)
		2001–2009	0.436 (1.19)	0.347 (0.98)	0.088 (3.21)

Table VII. Multivariate Fama-MacBeth Regressions, January 1966 to December 2009

Reported are results of implementing cross-sectional Fama-MacBeth regressions of monthly stock returns on NYSE-Amex stocks from 1964 to 2009, and also including Nasdaq stocks from 1983 to 2009. The coefficients reported in Column OLS, Column *RW* and Column DIF are as explained in the previous tables. *T*-statistics are reported in parentheses and adjusted for autocorrelation as in footnote 13 in Cooper, Gulen, and Schill (2008).

	OLS (*T*-stat.)	*RW* (*T*-stat.)	DIF (*T*-stat.)	OLS (*T*-stat.)	*RW* (*T*-stat.)	DIF (*T*-stat.)	OLS (*T*-stat.)	*RW* (*T*-stat.)	DIF (*T*-stat.)
		(1)			**(2)**			**(3)**	
Beta	0.059	0.006	0.052	0.038	0.007	0.030	0.082	0.019	0.063
	(0.48)	(0.05)	(2.82)	(0.35)	(0.07)	(2.59)	(0.76)	(0.20)	(3.91)
Size	-0.156	-0.097	-0.058	-0.060	-0.079	0.018	-0.177	-0.115	-0.062
	(-3.09)	(-1.96)	(-10.44)	(-1.70)	(-2.25)	(4.29)	(-1.90)	(-1.23)	(-4.35)
log(BM)	0.310	0.340	-0.030	0.311	0.341	-0.029	0.285	0.325	-0.040
	(3.93)	(4.21)	(-3.47)	(4.20)	(4.45)	(-3.00)	(4.10)	(4.49)	(-4.32)
InvPrice	–	–	–	0.178	0.005	0.173	–	–	–
				(1.54)	(0.05)	(8.96)			
Nydvol	–	–	–	–	–	–	0.131	0.098	0.033
							(1.18)	(0.85)	(1.53)
Nadvol	–	–	–	–	–	–	0.059	0.063	-0.004
							(0.30)	(0.31)	(-0.14)
Nyilliq	–	–	–	–	–	–	–	–	–
Nailliq	–	–	–	–	–	–	–	–	–
		(4)			**(5)**			**(6)**	
Beta	0.057	0.022	0.034	0.065	0.013	0.052	0.057	0.023	0.033
	(0.63)	(0.26)	(4.38)	(0.54)	(0.12)	(2.99)	(0.64)	(0.27)	(4.55)
Size	-0.092	-0.102	0.009	-0.106	-0.057	-0.048	-0.110	-0.117	0.006
	(-1.37)	(-1.48)	(0.88)	(-2.34)	(-1.28)	(-9.26)	(-1.66)	(-1.72)	(0.61)
log(BM)	0.288	0.325	-0.036	0.300	0.334	-0.034	0.283	0.321	-0.037
	(4.50)	(4.81)	(-3.40)	(3.82)	(4.14)	(-3.82)	(4.42)	(4.74)	(-3.45)
InvPrice	0.145	-0.022	0.168	–	–	–	0.054	-0.095	0.150
	(1.19)	(-0.19)	(8.49)				(0.46)	(-0.86)	(7.94)
Nydvol	0.109	0.078	0.030	–	–	–	0.130	0.103	0.027
	(0.89)	(0.63)	(1.11)				(1.08)	(0.86)	(0.98)
Nadvol	0.071	0.057	0.014	–	–	–	0.125	0.113	0.011
	(0.32)	(0.25)	(0.355)				(0.56)	(0.51)	(0.26)
Nyilliq	–	–	–	-0.013	-0.023	0.009	-0.025	-0.027	0.002
				(-0.15)	(-0.27)	(1.10)	(-0.24)	(-0.27)	(0.25)
Nailliq	–	–	–	0.089	0.082	0.006	0.088	0.087	0.001
				(3.15)	(2.90)	(1.46)	(3.32)	(3.24)	(0.19)

Internet Appendix for "Noisy Prices and Inference Regarding Returns"*

Supplement to Appendix B: Cross-sectional Implementation

Using the notation and definitions introduced in subsection B of Section II, in addition to introducing $D_{nt,s}$ to denote $\frac{1+\delta_{nt}}{1+\delta_{nt-s}}$, we can write the expressions for the probability limit of each (time t) cross-sectional estimator as follows.

A. Ordinary Least Squares, EW: $w_{nt} = \frac{1}{N}$

- $plim_{N\to\infty}\ \mu_{EW,t} = E(R^0_{nt}) = E(R_{nt}D_{nt}) = E(R_{nt}E(D_{nt}|\mathbf{X}, n))$, and
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}} = \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}R_{nt}D_{nt}\right) =$
$\left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}E(D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}$.

B. Weighting by the prior period's (gross) return, RW: $w_{nt} = R^0_{nt-1}$

- $plim_{N\to\infty}\ \mu_{RW,t} = \frac{E(R_{nt-1}R_{nt}D_{nt}D_{nt-1})}{E(R_{nt-1}D_{nt-1})} = \frac{E(R_{nt-1}R_{nt}E(D_{nt}D_{nt-1}|\mathbf{X},n))}{E\left(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}E(D_{nt-1}|\mathbf{X},n)\right)}$.
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}} = \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R^0_{nt-1}\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}R^0_{nt-1}R^0_{nt}\right) =$
$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}|\mathbf{X}, n)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}D_{nt}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}$.

C. Weighting by the prior s periods' cumulative (gross) return, RW(s):

$w_{nt} = R^0_{nt-1,s} = R^0_{t-1}R^0_{t-2}...R^0_{t-1-s}$.

As $D_{nt,s} = \frac{1+\delta_{nt}}{1+\delta_{nt-s}}$ (thus, $D_{nt,1} = D_{nt}$), then $R^0_{nt-1,s} = R_{nt-1,s}D_{nt-1,s}$ and

- $plim_{N\to\infty}\ \mu_{RW(s),t} = \frac{E(R_{nt-1,s}R_{nt}D_{nt}D_{nt-1,s})}{E(R_{nt-1,s}D_{nt-1,s})} = \frac{E(R_{nt-1,s}R_{nt}E(D_{nt,s+1}|\mathbf{X},n))}{E(R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n))}$.
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}} = \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}D_{nt-1,s})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}R_{nt-1,s}R_{nt}D_{nt-1,s}D_{nt}) =$
$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X}, n)\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt,s+1}|\mathbf{X}, n)\right)\tilde{\boldsymbol{\beta}}$.

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D. Weighting by the prior period's firm value, VW: $w_{nt} = S_n P^0_{nt-1}$

- $plim_{N\to\infty}\ \mu_{VW,t} = \frac{E(P_{nt-1}R_{nt}(1+\delta_{nt})S_n)}{E(P_{nt-1}(1+\delta_{nt-1})S_n)} = \frac{E(P_{nt}E(S_n(1+\delta_{nt})|\mathbf{X},n))}{E(P_{nt-1}E(S_n(1+\delta_{nt-1})|\mathbf{X},n))}$.
- $plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}} =$

$$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}(1+\delta_{nt-1})S_n\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}P_{nt-1}(1+\delta_{nt-1})S_nR_{nt}D_{nt}\right) =$$

$$= \left[E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt-1}|\mathbf{X},n)S_n\right)\right]^{-1} E\left(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt}|\mathbf{X},n)S_n\right)\tilde{\boldsymbol{\beta}}.$$

When $c = 0$ the following expressions, organized in a lemma, can be easily derived (using second-order Taylor approximations):

LEMMA B1:

1. $E(\delta_{nt}|\mathbf{X},n) = 0.$

2. $E\left(D_{nt}|\mathbf{X},n\right) \approx 1 + \sigma_n^2(1-\rho),$

3. $E(D_{nt}D_{nt-1}|\mathbf{X},n) \approx 1 + \sigma_n^2(1-\rho^2),$

4. $E(D_{nt,s}|\mathbf{X},n) \approx 1 + \sigma_n^2 - \sigma_n^2\rho^s.$

Proof:

1. Follows directly from the distributional assumptions for δ_{nt}.
2. $E\left(D_{nt}|\mathbf{X},n\right) \approx E\left((1+\delta_{nt})(1-\delta_{nt-1}+\delta^2_{nt-1})|\mathbf{X},n\right) =$
   $= E\left(1+\delta^2_{nt-1}-\delta_{nt}\delta_{nt-1}|\mathbf{X},n\right) = 1+\sigma_n^2(1-\rho).$
3. $E(D_{nt}D_{nt-1}|\mathbf{X},n) \approx E\left((1+\delta_{nt})(1-\delta_{nt-2}+\delta^2_{nt-2})|\mathbf{X},n\right) =$
   $= E\left(1+\delta^2_{nt-2}-\delta_{nt}\delta_{nt-2}|\mathbf{X},n\right) = 1+\sigma_n^2(1-\rho^2).$
4. $E(D_{nt,s}|\mathbf{X},n) \approx E\left((1+\delta_{nt})(1-\delta_{nt-s}+\delta^2_{nt-s})|\mathbf{X},n\right) =$
   $E\left(1+\delta^2_{nt-s}-\delta_{nt}\delta_{nt-s}|\mathbf{X},n\right) = 1+\sigma_n^2-\sigma_n^2\rho^s.$

Proof of Proposition 1:

$plim_{N\to\infty}\ \mu_{EW,t} = E\left(R_{nt}E(D_{nt}|\mathbf{X},n)\right) \approx E\left(R_{nt}\left(1+\sigma_n^2(1-\rho)\right)\right).$

$$plim_{N\to\infty}\ \mu_{RW,t} = \frac{E(R_{nt-1}R_{nt}E(D_{nt}D_{nt-1}|\mathbf{X},n))}{E(R_{nt-1}E(D_{nt-1}|\mathbf{X},n))} \approx \frac{E\Big(R_{nt-1}R_{nt}\big(1+\sigma_n^2(1-\rho^2)\big)\Big)}{E(R_{nt-1}(1+\sigma_n^2(1-\rho)))}.$$

$$plim_{N\to\infty}\ \mu_{RW(s),t} = \frac{E(R_{nt-1,s}R_{nt}E(D_{nt,s+1}|\mathbf{X},n))}{E(R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n))} \approx \frac{E\Big(R_{nt-1,s}R_{nt}(1+\sigma_n^2-\sigma_n^2\rho^{s+1})\Big)}{E(R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^s)}.$$

$$plim_{N\to\infty}\ \mu_{VW,t} = \frac{E(P_{nt}E(S_n(1+\delta_{nt})|\mathbf{X},n))}{E(P_{nt-1}E(S_n(1+\delta_{nt-1})|\mathbf{X},n))} = \frac{E(P_{nt}S_n)}{E(P_{nt-1}S_n)}.$$

Under the conditions of Proposition 1 the expressions in the propositions are a direct consequence of the expressions developed in Lemma B1.

Proof of Proposition 2:

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}} = \Big[E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\Big)\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}E(D_{nt}|\mathbf{X},n)\Big)\tilde{\boldsymbol{\beta}} \approx$$
$$\Big[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\left(1+\sigma_n^2(1-\rho)\right)\Big)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}} =$$
$$= \Big[E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}|\mathbf{X},n)\Big)\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}E(D_{nt-1}D_{nt}|\mathbf{X},n)\Big)\tilde{\boldsymbol{\beta}} \approx$$
$$\Big[E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}\left(1+\sigma_n^2(1-\rho)\right)\Big)\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1}\left(1+\sigma_n^2(1-\rho^2)\right)\Big)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}} =$$
$$= \Big[E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt-1,s}|\mathbf{X},n)\Big)\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}E(D_{nt,s+1}|\mathbf{X},n)\Big)\tilde{\boldsymbol{\beta}} \approx$$
$$\Big[E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^s)\Big)\Big]^{-1} E\Big(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}R_{nt-1,s}(1+\sigma_n^2-\sigma_n^2\rho^{s+1})\Big)\tilde{\boldsymbol{\beta}}.$$

$$plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}} =$$
$$= \Big[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt-1}|\mathbf{X},n)S_n)\Big]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}E(1+\delta_{nt}|\mathbf{X},n)S_n)\tilde{\boldsymbol{\beta}} =$$
$$= \Big[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}S_n)\Big]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}P_{nt-1}S_n)\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}.$$

Under the conditions of Proposition 2 the expressions in the propositions are a direct consequence of the expressions above and those developed in Lemma B1.

The following expressions, organized in a Lemma, will be used to proof Propositions 3 and 4.

LEMMA B2:

1. $\bar{E}\left(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})}\right) = 1+\sigma^2 c(1-\rho)$,

2. $\bar{E}(E(D_{nt})) \approx \bar{E}(E(1+\delta_{nt}-\delta_{nt-1}-\delta_{nt}\delta_{nt-1}+\delta^2_{nt-1}+\delta_{nt}\delta^2_{nt-1})) = 1+\sigma^2(1-\rho)$,

3. $\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = 1+\sigma^2(1-\rho)(\rho-c\rho+c)$,

4. $\bar{E}(\frac{E(D_{nt-s}D_{nt})}{E(D_{nt-1})}) = 1+\sigma^2(1-\rho)\rho^{s-1}+\sigma^2 c(1-2\rho^{s-1}+\rho^s)$.

Proof:

1. This follows directly from the distributional assumptions on δ_{nt}.

2. $\bar{E}(E(D_{nt})) \approx \bar{E}(E(1+\delta_{nt}-\delta_{nt-1}-\delta_{nt}\delta_{nt-1}+\delta^2_{nt-1}+\delta_{nt}\delta^2_{nt-1})) = 1+\bar{E}(\sigma^2_n(-\delta^0_{nt}\delta^0_{nt-1}+(\delta^0_{nt-1})^2+\delta^0_{nt}(\delta^0_{nt-1})^2)) = 1+\sigma^2(1-\rho)$.

3. $\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = \bar{E}(\frac{E((1+\delta_{nt})(1-\delta_{nt-2}+\delta^2_{nt-2}))}{E((1+\delta_{nt-1})(1-\delta_{nt-2}+\delta^2_{nt-2}))}) = \bar{E}(\frac{E(1-\delta_{nt-2}+\delta^2_{nt-2}+\delta_{nt}-\delta_{nt}\delta_{nt-2}+\delta_{nt}\delta^2_{nt-2})}{E(1-\delta_{nt-2}+\delta^2_{nt-2}+\delta_{nt-1}-\delta_{nt-1}\delta_{nt-2}+\delta_{nt-1}\delta^2_{nt-2})})$.
   Use a to denote $E(-\delta_{nt-2}+\delta^2_{nt-2}+\delta_{nt}-\delta_{nt}\delta_{nt-2}+\delta_{nt}\delta^2_{nt-2})$ and b to denote $E(-\delta_{nt-2}+\delta^2_{nt-2}+\delta_{nt-1}-\delta_{nt-1}\delta_{nt-2}+\delta_{nt-1}\delta^2_{nt-2})$.
   Then $\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = \bar{E}(\frac{1+a}{1+b}) \approx \bar{E}((1+a)(1-b+b^2)) = \bar{E}(1+a-b-ab+b^2+ab^2)$.
   From here tedious but straight-forward calculations[1] using only the definition of δ^0_{nt} lead to:
   $\bar{E}(a) = \sigma^2(1-\rho^2)$, $\bar{E}(b) = \sigma^2(1-\rho)$, and $\bar{E}(ab) \approx \sigma^2 c(1-\rho^2)$. In this approximation we ignore all terms of order σ^4 and higher.
   In addition, $\bar{E}(b^2) = 2\sigma^2 c(1-\rho)$ and $\bar{E}(ab^2) \approx 0$.

4. $\bar{E}(\frac{E(D_{nt-s}D_{nt})}{E(D_{nt-1})}) = \bar{E}(\frac{E((1+\delta_{nt})(1-\delta_{nt-s}+\delta^2_{nt-s}))}{E((1+\delta_{nt-1})(1-\delta_{nt-s}+\delta^2_{nt-s}))}) = \bar{E}(\frac{E(1-\delta_{nt-s}+\delta^2_{nt-s}+\delta_{nt}-\delta_{nt}\delta_{nt-s}+\delta_{nt}\delta^2_{nt-s})}{E(1-\delta_{nt-s}+\delta^2_{nt-s}+\delta_{nt-1}-\delta_{nt-1}\delta_{nt-s}+\delta_{nt-1}\delta^2_{nt-s})})$.
   As above, let c denote$E(-\delta_{nt-s}+\delta^2_{nt-s}+\delta_{nt}-\delta_{nt}\delta_{nt-s}+\delta_{nt}\delta^2_{nt-s})$, and let d denote $E(-\delta_{nt-s}+\delta^2_{nt-s}+\delta_{nt-1}-\delta_{nt-1}\delta_{nt-s}+\delta_{nt-1}\delta^2_{nt-s})$. Then $\bar{E}(\frac{E(D_{nt-s}D_{nt})}{E(D_{nt-1})}) = \bar{E}(\frac{1+c}{1+d}) \approx \bar{E}((1+c)(1-d+d^2)) = \bar{E}(1+c-d-cd+d^2+cd^2)$. From here $\bar{E}(c) = \sigma^2(1-\rho^s)$, $\bar{E}(d) = \sigma^2(1-\rho^{s-1})$. $\bar{E}(cd) \approx \sigma^2 c(1-\rho^s)$ (in this approximation we ignore all terms of order σ^4 and higher), $\bar{E}(d^2) = 2\sigma^2 c(1-\rho^{s-1})$, and $\bar{E}(cd^2) \approx 0$.

Proof of Proposition 3:

Substitute the expressions from Lemma B2 into the expressions below to get the expressions in the proposition.

$\bar{E}(plim_{N\to\infty}\ \mu_{EW,t}) = \bar{E}(E(R^0_{nt})) = \bar{E}(E(R_{nt}D_{nt})) = \bar{E}(E(R_{nt}E(D_{nt}|\mathbf{X}, n)))$.

$\bar{E}(plim_{N\to\infty}\ \mu_{RW,t}) = \bar{E}(\frac{E(R_{nt-1}R_{nt}D_{nt}D_{nt-1})}{E(R_{nt-1}D_{nt-1})}) = \frac{E(R_{nt-1}R_{nt})}{E(R_{nt-1})}\bar{E}\frac{E(D_{nt}D_{nt-1})}{E(D_{nt-1})}$.

$\bar{E}(plim_{N\to\infty}\ \mu_{RW(s),t}) = \bar{E}(\frac{E(R_{nt-1,s}R_{nt}D_{nt}D_{nt-1,s})}{E(R_{nt-1,s}D_{nt-1,s})}) = \frac{E(R_{nt-1,s}R_{nt})}{E(R_{nt-1,s})}\bar{E}\frac{E(D_{nt}D_{nt-1,s})}{E(D_{nt-1,s})}$.

$\bar{E}(plim_{N\to\infty}\ \mu_{VW}, t) = \bar{E}(\frac{E(P_{nt-1}R_{nt}(1+\delta_{nt})S_n)}{E(P_{nt-1}(1+\delta_{nt-1})S_n)}) = \frac{E(P_{nt-1}R_{nt}S_n)}{E(P_{nt-1}S_n)}\bar{E}(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})})$.

Proof of Proposition 4:

$$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{EW,t}}) = \bar{E}(\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}D_{nt})\tilde{\boldsymbol{\beta}}) =$$
$$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} \bar{E}(E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}D_{nt})\tilde{\boldsymbol{\beta}})) = \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}\bar{E}(D_{nt})\tilde{\boldsymbol{\beta}})) =$$
$$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(1+\sigma_n^2(1-\rho))\tilde{\boldsymbol{\beta}})).$$

$$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW,t}}) =$$
$$= \bar{E}\{\left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}})D_{nt-1})\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}})D_{nt-1}D_{nt})\tilde{\boldsymbol{\beta}}\} =$$
$$= \left[E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}))\right]^{-1} E(\tilde{\mathbf{X}}'_{\mathbf{nt}}\tilde{\mathbf{X}}_{\mathbf{nt}}(\tilde{\mathbf{X}}_{\mathbf{nt-1}}\tilde{\boldsymbol{\beta}}))\tilde{\boldsymbol{\beta}}\}\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}) = \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(D_{nt-1}D_{nt})}{E(D_{nt-1})}).$$

$$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{RW(s),t}}) \approx \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(D_{nt-1,s}D_{nt})}{E(D_{nt-1,s})}).$$

$$\bar{E}(plim_{N\to\infty}\tilde{\boldsymbol{\beta}}_{\mathbf{VW,t}}) \approx \tilde{\boldsymbol{\beta}}\bar{E}(\frac{E(1+\delta_{nt})}{E(1+\delta_{nt-1})}).$$

Appendix C: Evaluating the Proposed Corrections in the Presence of Time Variation in Discount Rates

In this appendix we consider the performance of the RW, IEW, and VW corrections in the presence of stationary time-varying discount rates. As noted in Section II, the distinguishing characteristic of noise in transaction prices is that noise is temporary, and is reversed over time. However, not all temporary components in prices necessarily reflect noise. Poterba and Summers (1988), among others, have observed that time variation in required returns, i.e., in the discount rates used to assess the present value of dividends, induces a transitory component in prices. To the extent that such variation in discount rates reflects variation in economic fundamentals such as risk aversion, volatility, investment opportunities, etc., a temporary component is induced in true prices. Time variation in discount rates is empirically important: Cochrane (2011) reports that as much as half of time variation in market returns (and essentially all of the time variation in price-dividend ratios) can be attributed to discount-rate variation.

Of course, discount-rate variation need not be rational: Cochrane (2011, p. 1067) observes that "Behavioral theories are also discount-rate theories." Economic models are required to ascertain the extent to which the observed variation in discount rates is consistent with rational vs. behavioral explanations. Such evaluation is beyond the scope of this paper. We note that, to the extent that discount-rate variation reflects behavioral sources the resulting temporary component in prices comprises noise, for which we evaluate corrections. In contrast, to the extent that the discount-rate variation reflects economic fundamentals the resulting temporary component in prices should not be interpreted as noise, and requires no correction. We assess in this appendix the potential effect of time variation in discount rates on the properties of the RW, IEW, and VW estimators.

To be conservative, we assume for this analysis that all variation in discount rates is due to economic fundamentals, and none is due to behavioral explanations. Also, to assess the maximum feasible effect, we also assume that all variation in true market returns is attributable to variation in discount rates, i.e. that none of the variation in market returns is due to changing expectations regarding dividends. We first develop closed form solutions for the effect of the RW correction on time-series and cross-sectional mean returns in a simplified setting where prices contain no noise. We then modify the simulations described earlier to allow for time varying market discount rates, and assess the resulting effects on the properties of the corrected estimates of means and regression coefficients in a more realistic setting. We rely on the return decomposition of Campbell and Shiller (1988), and the extensions thereof presented by Campbell (1991, 2001).

A. Closed Form Solutions in a Simple Setting

We first consider what might be viewed as a "worst-case" scenario. There is no noise in prices, implying that no corrections are required. However, there is a temporary component in market prices, due to stationary time-varying discount rates. Further, we make the extreme and unrealistic assumption that *all* variation in ex post market returns is attributable to rational changes in discount rates, and none is due to variations in dividend expectations or behavioral explanations. A researcher mistakenly implements the *RW* correction for noise when calculating mean returns. How much damage is done?

Following Campbell (2001), the expected stock return, x_t, is assumed to follow an AR(1) process,

$$x_{t+1} = \phi x_t + u_{t+1}, \quad -1 < \phi < 1. \tag{C-1}$$

If ϕ is zero discount rates vary randomly, while as ϕ increases discount-rate changes become more persistent. Relying on the log-linear return decomposition of Campbell and Shiller (1988), realized (log) returns can be expressed as (Campbell, 2001, expression 2.5):

$$r_{t+1} = x_t + v_{d,t+1} - \frac{\kappa u_{t+1}}{1 - \kappa\phi}, \tag{C-2}$$

where $v_{d,t+1}$ is the change from t to $t+1$ in the expectation of the discounted (at rate κ) value of future dividends. The factor κ is approximately the inverse of 1 plus the average dividend yield.

The variance of log returns is given by expression (2.6) in Campbell (2001) as:

$$\sigma_r^2 = \frac{1 - 2\kappa\phi + \kappa^2}{(1-\kappa\phi)^2}\sigma_x^2 + \sigma_d^2 - \frac{2\kappa}{1-\kappa\phi}\sigma_{dx}, \tag{C-3}$$

where $\sigma_x^2 = Var(x_{t+1})$, $\sigma_d^2 = Var(v_{d,t+1})$, and $\sigma_{dx} = Cov(v_{d,t+1}, x_{t+1})$, which we rewrite as:

$$\sigma_r^2 = A\sigma_x^2 + \sigma_d^2 - B\sigma_{dx}, \tag{C-4}$$

where $A = \frac{1-2\kappa\phi+\kappa^2}{(1-\kappa\phi)^2}$ and $B = \frac{2\kappa}{1-\kappa\phi}$.

We follow Campbell (2001) in using the approximation $\kappa = 1$,[2] in which case A = B, and the first-order serial covariance of returns is given by expression (2.9) in Campbell (2001), as:

$$cov(r_{t-1}, r_t) = \sigma_{dx} - \sigma_x^2. \tag{C-5}$$

The preceding expressions pertain to log returns, while our analysis considers simple gross returns. However, a first-order Taylor series approximation establishes that variance and first-order serial covariance of simple gross returns are approximately equal to their log return counterparts.

To place an upper bound on the potential damage caused by implementing the RW correction we impose the extreme assumption that *all* variation in realized returns is caused by variation in discount rates (and thus none of the variation in returns is attributable to variation in dividend expectations or noise). In particular, we set $\sigma_d^2 = \sigma_{dx} = 0$, which implies $\sigma_x^2 = \frac{\sigma_R^2}{A}$, where $A = \frac{2}{1-\phi}$, or

$$\sigma_x^2 = \frac{\sigma_R^2(1-\phi)}{2}. \tag{C-6}$$

Time-Series Mean Returns

The bias in an estimate of the series mean return obtained by use of RW estimator is given by

$$plim\ \mu_{RW} - \mu = \frac{E(R_{t-1}R_t)}{E(R_{t-1})} - E(R_t) = \frac{Cov(R_{t-1}, R_t)}{E(R_{t-1})}, \tag{C-7}$$

where expectations and covariances are in the time series. Relying on equation (C-6), we have in this extreme case

$$plim\ \mu_{RW} - \mu = -\frac{\sigma_R^2(1-\phi)}{2E(R_t)}. \tag{C-8}$$

The bias in the time-series mean return induced by the RW correction is negative, and is largest in absolute magnitude when $\phi = 0$. Even in this extreme case the bias is modest. For example, if the return standard deviation is 4.5% and the expected gross return is 1.009 the downward bias is just 10 basis points per month.[3] In the more realistic case where expected returns are persistent, the bias is mitigated. For example, Campbell (1991, page 168) indicates that the empirical evidence is consistent with ϕ for monthly returns approximately equal to 0.8. If so, the downward bias is reduced in this case to 2 basis points per month, despite the fact that we continue to impose the unrealistic assumption that all variation in returns is attributable to rational discount-rate variation.

Cross-sectional Mean Returns

We next assess the bias introduced to the estimated cross-sectional mean return when the RW correction is implemented even though it is not required. We consider the case where the expected return to each security is linked to the overall market return by constant, security specific beta coefficients, and *all* variation in market returns is due to rational changes in discount rates. In particular,

$$E(R_{nt}|n, R_{mt}) = 1 + \alpha + \beta_n(R_{mt} - 1). \tag{C-9}$$

Here, expectations are cross-sectional, and are conditional on the time-t market return, implying that

$$E(R_{nt}) = R_{mt}. \tag{C-10}$$

The *plim* of the bias in period t is $plim_{N\to\infty}\ \mu_{RW,t} - \mu = \frac{E(R_{nt-1}R_{nt})}{E(R_{nt-1})} - E(R_{nt}) = \frac{Cov(R_{nt-1},R_{nt})}{E(R_{nt-1})}$, where expectations and covariances are cross-sectional, and are conditional on the time-t market return.

We will evaluate the mean (across t) bias, or $\bar{E}(\frac{Cov(R_{nt-1},R_{nt})}{E(R_{nt-1})})$, where the bar indicates that the expectation is taken in the time series. We thus have:
$\frac{Cov(R_{nt-1},R_{nt})}{E(R_{nt-1})} = \frac{Cov(\beta_n(R_{mt-1}-1),\beta_n(R_{mt}-1))}{R_{mt-1}} = \frac{\sigma^2_\beta(R_{mt-1}-1)(R_{mt}-1)}{R_{mt-1}} = \sigma^2_\beta(R_{mt}-1+\frac{1-R_{mt}}{R_{mt-1}}) \approx$

$$\approx \sigma^2_\beta\left(R_{mt} - 1 + (1 - R_{mt})(2 - R_{mt-1})\right), \tag{C-11}$$

where σ^2_β is the cross-sectional variance of security beta coefficients.[4]

It follows that

$$\begin{aligned}\bar{E}(bias) &= \bar{E}(\frac{Cov(R_{nt-1}, R_{nt})}{E(R_{nt-1})}) \approx \sigma^2_\beta\bar{E}(R_{mt} - 1 + (1 - R_{mt})(2 - R_{mt-1}))\\ &= \sigma^2_\beta\left(1 - 2\bar{E}(R_{mt}) + \bar{E}(R_{mt}R_{mt-1})\right)\sigma^2_\beta(1 - 2\bar{E}(R_{mt}) + Cov(R_{mt}, R_{mt-1}) + \bar{E}(R_{mt})^2)\\ &= \sigma^2_\beta\left((\bar{E}(R_{mt}) - 1)^2 + Cov(R_{mt}, R_{mt-1})\right). \end{aligned} \tag{C-12}$$

Imposing the condition $\sigma^2_d = \sigma_{dx} = 0$ for market returns, we have $Cov(R_{mt}, R_{mt-1}) = -\frac{\sigma^2_{R_m}(1-\phi)}{2}$ and as a result

$$\bar{E}(bias) = \sigma^2_\beta\left(\left(\bar{E}(R_m) - 1\right)^2 - \frac{\sigma^2_{R_m}(1 - \phi)}{2}\right). \tag{C-13}$$

Notably, there is no bias in the cross-sectional mean return if all betas are equal. The bias, which depends also on the mean market return, the variance of market returns, and the persistence in market discount rates ϕ, is positive when $\phi = 1$ (implying that discount rates are not mean reverting) and for reasonable market parameters is negative when ϕ is small. In any case, the induced bias is minuscule. For example, if the cross-sectional standard deviation of beta is 0.5, the standard deviation of market returns is 4.5%, and the expected net market return is 0.9%, the bias ranges from -2 basis points when $\phi =$ 0 to 2 basis points when $\phi = 1$. We emphasize again that the bias is minuscule though we assume that all variation in market returns is attributable to rational discount-rate changes, and none is due to changes in dividend expectations or mispricing.

B. Simulation Analysis in a More Complex Setting

The preceding analysis demonstrates, in a simplified setting, that the "damage" imposed by implementing the *RW* correction when true prices contain a temporary component due to variation in discount rates is minuscule. For robustness, we assess through an extension of the simulations reported in Section II.D above the generality of this conclusion in a more realistic setting. In particular, we allow for noise in observed prices, for cross-sectional variation in mean returns related to market beta, illiquidity, and firm size, and for non-zero correlations between noise variance and firm characteristics. Parameters of the simulation are identical to those described in Section II.D above, except that we replace the independently distributed true market returns with market returns that reflect time variation in discount rates, as in expression (A-2) above. We extend the analysis to the VW and IEW corrections as well, and consider the effect on cross-sectional regression slope coefficients as well as mean returns.

Also, since the intent here is to rely on parameters that are as realistic as possible, we do not incorporate the extreme assumption that all variation in market returns is attributable to changes in discount rates. Instead, we incorporate the empirical observation of Campbell (1991, page 168) that "slightly more than a third of the variance of unexpected returns is attributed to the variance of news about future cash flows, slightly less than a third is attributed to the variance of news about future returns, and the remainder is due to the covariance term." Thus, we allocate one third of the stock return variance to each of the three terms in expression C-3, implying:

$$\sigma_d^2 = \frac{\sigma_r^2}{3} \tag{C-14}$$

$$\sigma_x^2 = \frac{\sigma_r^2}{3A} \tag{C-15}$$

$$\sigma_{dx} = \frac{-\sigma_r^2}{3B} = \frac{-\sigma_d^2}{B} \tag{C-16}$$

Campbell (2001) also notes that:

$$\sigma_{dx} = \sigma_{du} \tag{C-17}$$

and

$$\sigma_u^2 = (1 - \phi^2)\sigma_x^2. \tag{C-18}$$

To impose the correct covariance in our simulations, we use expression (C-16) and (C-17) and define u as a linear projection on v_d:

$$u = \gamma v_d + \omega, \tag{C-19}$$

where

$$\gamma = \frac{\sigma_{du}}{\sigma_d^2} = \frac{\sigma_{dx}}{\sigma_d^2} = \frac{-1}{B}. \tag{C-20}$$

From expression (C-19) we have $\sigma_u^2 = \gamma^2\sigma_d^2 + \sigma_\omega^2$. Substituting from expressions (C-14), (C-15), (C-17), and (C-18) we have:

$$\sigma_\omega^2 = \left(\frac{1-\phi^2}{A} - \frac{1}{B^2}\right)\frac{\sigma_r^2}{3}. \tag{C-21}$$

To be consistent with the prior simulations σ_r^2 is set as 0.055^2. The simulated true market returns are created as follows: we specify σ_d^2 according to expression (C-14), σ_ω^2 according to expression (C-21), and $\gamma = \frac{-1}{B}$ according to expression (C-20). We create simulated outcomes assuming zero-mean normal distributions for v_d and ω, define u according to expression (A-12) and x by equation (C-1), and then compute log returns according to equation (C-2). Log returns are converted to equivalent simple returns, and we add the constant .01 - $\sigma_r^2/2$ to ensure that the simulated simple returns have a mean of .01, as in the earlier simulations. We do so for various possible levels of the ϕ parameter. Having done so, we conduct simulations identical to those described in Section II.D, except for the use of true market returns that reflect time variation in discount rates as described.

Simulation Results

We first discuss the simulation results with regard to the estimation of the cross-sectional mean return. Figure C1(a) displays the average (across 30,000 replications) of the bias in the estimated RW, VW, IEW, and the OLS mean returns, as a function of the parameter ϕ ranging from 0 to 0.9, when the parameters C are ρ are both 0. Figure C1(b) displays corresponding results when $\rho = c = 0.5$. To highlight the incremental effect of allowing for time variation in discount rates, Figures C1(c) and C1(d) display the differences between the biases obtained here and the biases that obtained in corresponding simulations with independent true market returns (as presented in section II.D).

<**Figure C1 about here**>

As evident from the figures, the main determinant of the bias in the weighted estimators is the values of the ρ and c parameters that pertain to the behavior of the noise in prices, while, as in the theoretical development in the simplified setting, the effect of time variation in discount rates is very small. In particular, when $\rho = c = 0$, the RW bias is in the range of -1.2 to 1.2 basis points (depending on ϕ), and the RW bias differs by at most 2.5 basis points from the corresponding bias when discount rates are constant. The corresponding ranges for the biases in the VW and IEW estimators are -7 to -5 and 4 to 5 basis points. The differential bias as compared to the corresponding simulations without time varying discount rates are uniformly less than 3 basis points for both estimators.

When $\rho = c = 0.5$, the RW bias ranges between 14 and 17 basis points. The corresponding bias range for the VW estimator is 6 to 7.5 basis points, and for the IEW estimator is 17 to 19 basis points. However, all three estimators showed similar bias when market returns were independent. For all three estimators, the differential bias as compared to the corresponding cases where discount rates are constant is at most 3 basis points. Thus, the simulations confirm that the effect of time-varying discount rates on corrected estimates of the cross-sectional mean return are monotone in ϕ, and are very small in magnitude, even in a more complex and realistic setting.

Finally, we assess the properties of the bias in the regression slope coefficients estimated from the simulated observed returns. As before, we focus our discussion on the estimation of the return premium on *illiq*. Figures C1(e) and C1(f) display the average bias in slope coefficients obtained by the different weighting method, for the ϕ parameter ranging from 0 to 0.9, for $\rho = c = 0$ and $\rho = c = 0.5$ correspondingly. For all practical purposes, the slopes are not affected by time variation in discount rates, and the differences in biases obtained here and corresponding biases with constant discount rates as presented in Section II.D is essentially zero.

In summary, the corrections proposed here remain effective in the presence of time variation in market discount rates. The marginal impact of time variation in market discount rates on the bias in the corrected estimators is generally two basis points or less. By comparison, the empirical effect of implementing the corrections reported here are much larger in absolute magnitude. We conclude that time variation in market discount rates does not appreciably affect the performance of the estimators proposed and has little to no impact on the empirical results reported here.

Footnotes

[1]The calculations are available from the authors upon request.

[2]Campbell, Lo, and MacKinlay (1996, Chapter 7) note that empirically in US data over the period 1926 to 1994 the average dividend-price ratio has been about 4% annually, implying that κ should be about 0.96 in annual data, or about 0.997 in monthly data.

[3]The mean and standard deviation noted are in line with outcomes for CRSP VW returns from 1963 to 2009. While VW index returns potentially contain noise the effect on the mean return is likely to be small. Further, to the extent that the standard deviation of observed VW returns exceeds that of true VW returns, the effect is to overstate the potential biases attributable to time-varying discount rates.

[4]This relies on the first-order Taylor series approximation $\frac{1}{R_{mt-1}} \approx 2 - R_{mt-1}$

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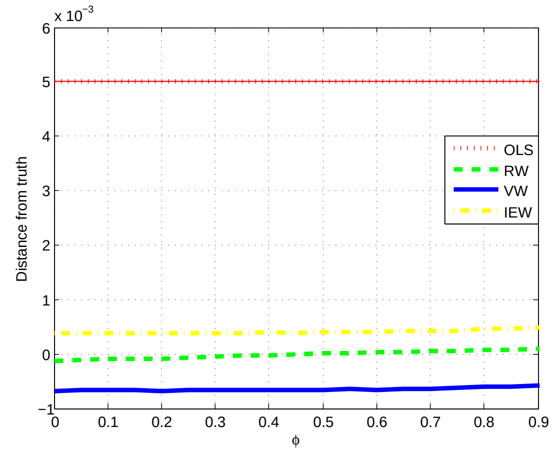
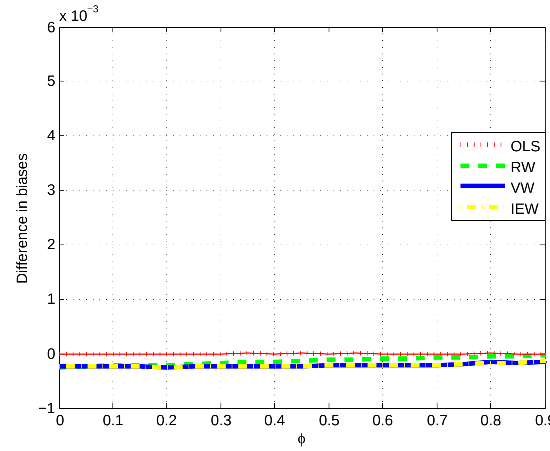
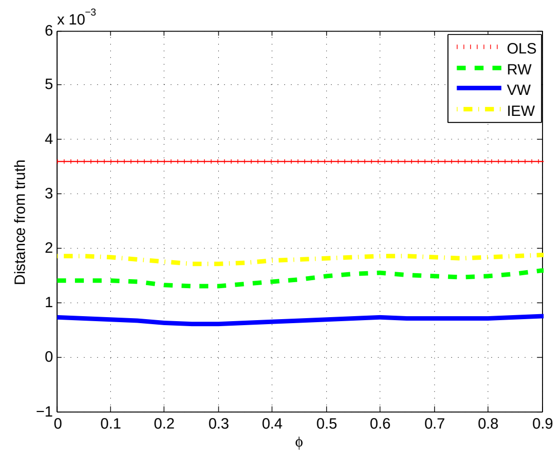
Figure C1. Simulation Results: The Effect of Time-Varying Discount Rates



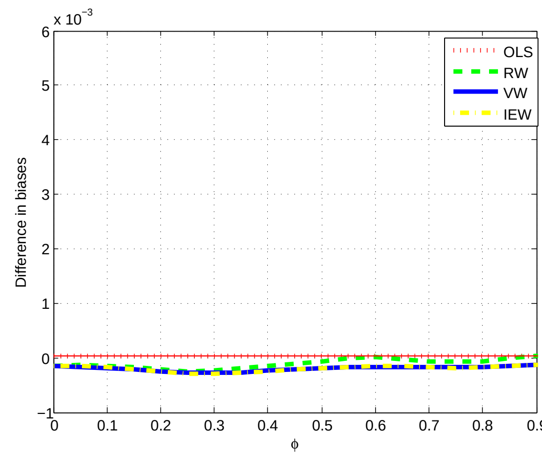
(a) Mean bias when $c = \rho = 0$.



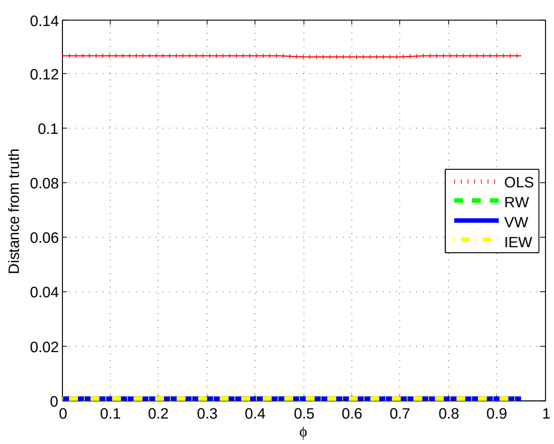
(b) Mean bias difference in time-varying vs. constant discount rates, $c = \rho = 0$.



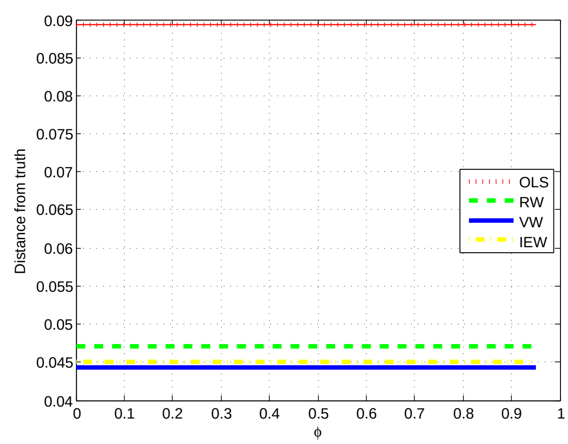
(c) Mean bias when $c = \rho = 0.5$.



(d) Mean bias difference in time-varying vs. constant discount rates, $c = \rho = 0.5$.



(e) Slope bias when $c = \rho = 0$.



(f) Slope bias when $c = \rho = 0.5$.